Trigonometric Solution

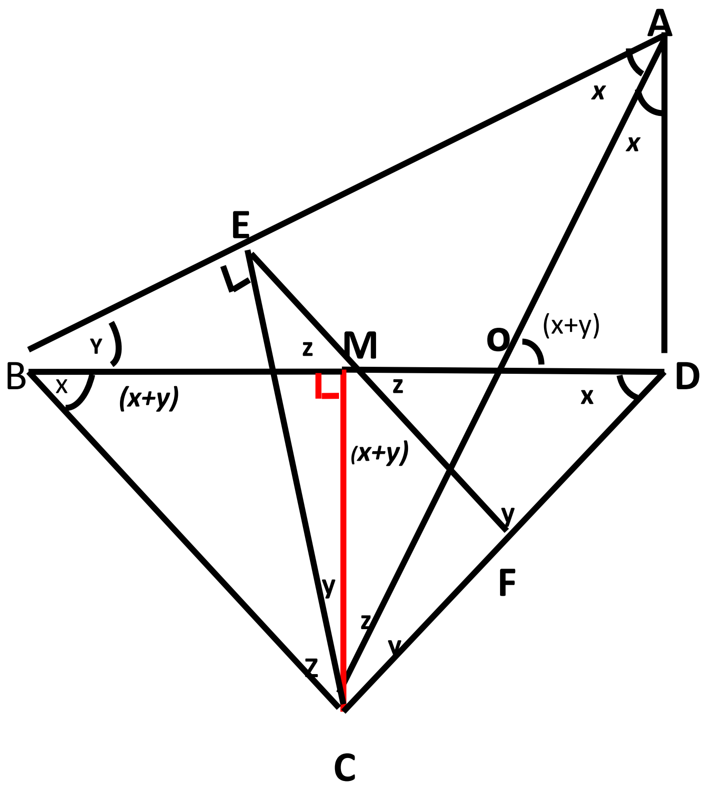
As a Geometrician, I didn't approve of trigonometric solutions for Geometric problems. But it's academically interesting to develop/study trigonometric solutions confirming Geometric results. For this problem also, I developed a Trigonometric Solution which I am happy to share here.

- Author

Question:

ABCD is a quadrilateral with BC = CD and AB > AD. The diagonal AC is also the internal bisector of $\angle BAD$. CE is drawn perpendicular to AB. M is the midpoint of BD. EM is joined and produced to meet CD at F.

Prove that BM $= \frac{\text{CE x FM}}{\text{FC}}$



Solution : 2

In Δ ABC,

$\frac{BC}{\sin x} = \frac{AC}{\sin\angle \text{ABC}}$ ....(1)

In Δ ADC, $\frac{CD}{\sin x} = \frac{AC}{\sin ADC}$ ....(2)

BC = CD (given) ....(3)

(1), (2) & (3) $\rightarrow \sin\angle \text{ABC} = \sin\angle \text{ADC}$

∴ $\angle$ ABC = $\angle$ ADC (or) = 180 - $\angle$ ADC

If $\angle ABC = \angle ADC$, then $\Delta ABC \cong \Delta ADC$ & AB = AD

But it is given that AB > AD

$\therefore \angle ABC = \angle ADC$ is inapplicable.

$\therefore \angle ABC = 180 - \angle ADC$

$\therefore$ ABCD is concyclic.

Now join CM.

Let $\angle BAC = \angle DAC = x$

Let $\angle ABD = y$

$\therefore \angle ACD = y$

$\therefore \angle CBD = \angle CDB = x$

$\angle CBE = (x + y)$ ....(4)

$\angle CMB = \angle CMD = 90^\circ$ ....(5)

$\angle CEB = 90^\circ$ (given) ....(6)

$\therefore EBCM$ is concyclic.

$\angle EBM = \angle ECM = y$

$\angle EBC = \angle FMC = (x + y)$ ....(7)

$\angle AOD = (x + y) = \angle MOC$ ....(8)

(7)& (8) $\rightarrow \Delta CBE \sim \Delta COM$

$\therefore \angle MCO = \angle ECB = z$ (say) ....(9)

$\angle MCF = (z + y)$ ....(10)

$\therefore$ In Δ MCF,

$\frac{MF}{FC} = \frac{\sin(z+y)}{\sin(x+y)}$ ....(11)

In Δ BCM, $sin\ (z + y) = \frac{BM}{BC}$ ....(12)

In Δ BCE, $sin\ (x + y) = \frac{CE}{BC}$ ....(13)

(11), (12) & (13) $\rightarrow \frac{MF}{FC} = \frac{BM}{CE}$

$\Longrightarrow BM = CE \times \frac{MF}{FC}$ proved.