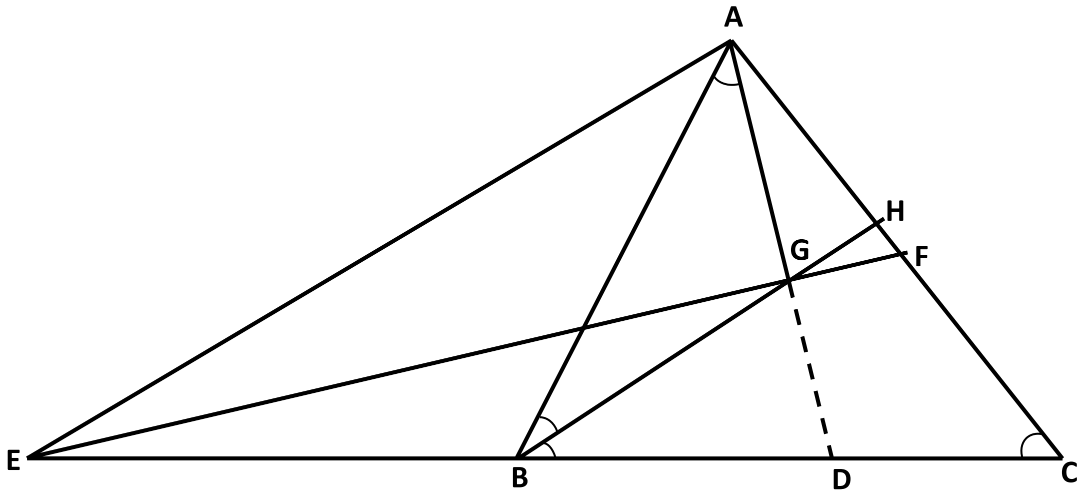
Given:

In Δ ABC, BH is the bisector of $\angle$B meeting AC at H. CB is produced to E such that AB =BE. F is the midpoint of AC. EF and BH meet at G and AG is joined.

To prove: AG = AH



Proof:

Produce AG to meet BC at D.

Now, as per Angle Bisector theorem.

$$\frac{AG}{GD}=\frac{AB}{BD} \quad \text{--} \quad (1)$$

(Angle Bisector theorem is my favourite theorem. So, first I wrote this step. Next, as I am always fond of the Menelaus theorem, and there also I will get $\frac{AG}{GD}$, I wanted to employ it here.)

For ΔADC, FGE is a transversal.

As per Menelaus Theorem,

$$\therefore \frac{CF}{FA}\times\frac{AG}{GD}\times\frac{DE}{EC}=1$$

Since $\frac{CF}{FA}=1$,

$$\frac{AG}{GD}=\frac{EC}{DE}$$

(From this, I got $\frac{AB}{BD}=\frac{EC}{DE}$. Upto this point, I developed the proof on the very first day itself. But beyond this point, I couln't proceed. In fact, I was clueless. So, I dropped

my idea of using the above theorems. I changed my strategy. I made some new constructions. I joined CG and produced it to meet AE at X. I joined XD. I proved XD parallel AC and got some similar triangles and tried many methods but I couldn't get anywhere near the proof. I became vexed and gave up my efforts.)

(Then the lockdown came. I remembered this problem again. One morning, I made special prayers and reopened my pursuit. This time, I was determined not to resort to the method of producing GC to X and so on. I went back to Angle Bisector Theorem and Menelaus Theorem and tried my luck again.)

ie $\frac{AG}{GD} = \frac{AB+BC}{AB+BD}$ (Because AB=EB) ---------------------------- 2

1 & 2 $\rightarrow$

$$\frac{AB}{BD} = \frac{AB+BC}{AB+BD}$$

(I felt, I was zeroing in)

Since, each ratio is $= \frac{difference\ between\ numerators}{difference\ between\ denominators}$

$$\frac{AB}{BD} = \frac{AB+BC-AB}{AB+BD-BD}$$

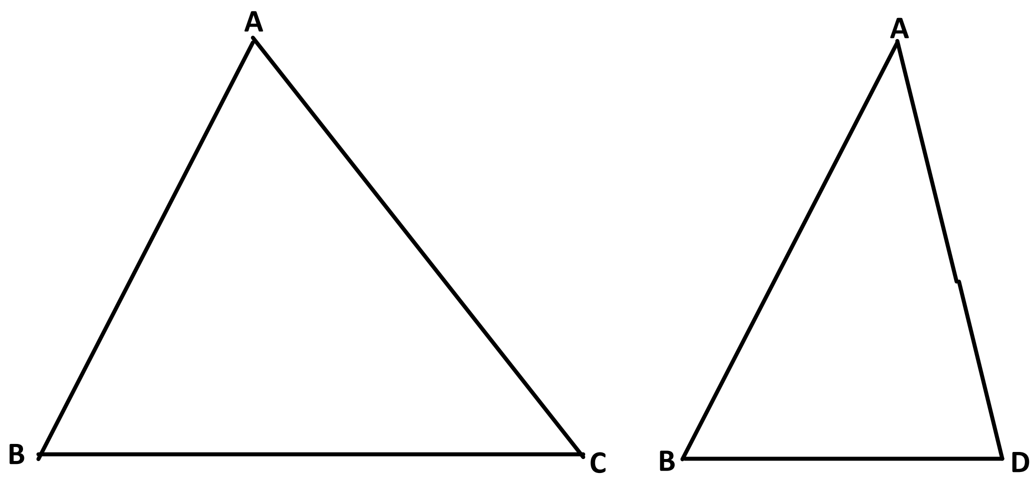
$$= \frac{BC}{AB}$$

$$\frac{AB}{BD} = \frac{BC}{AB} \text{ --------- } 3$$

(I got this result. I felt that the target was somewhat nearby and keenly watched the figure for about two minutes and then I noticed the common $\angle ABD$ in Δ ABC and Δ ABD. I became excited and prowled for similarity between these two triangles. Then I remembered the SAS principle and clinched my catch.)

Now, consider the following two triangles

Δ ABC and Δ DBA



3 $\longrightarrow$ $$\frac{AB}{BD} = \frac{BC}{AB}$$

$\angle ABD = \angle ABC$

3 $\longrightarrow$ $$\frac{AB}{BD} = \frac{BC}{AB}$$

In two triangles, if the ratio between two corresponding sides are equal and the angles between those two sides are also equal, then, those two triangles are similar. (SAS principle). Therefore these two triangles are similar.

$\therefore$ These triangles are similar.

$\angle BAG = \angle C$ -------------------- (4)

(Hereafter, it was just a cakewalk to prove that AG = AH)

Now, $\angle AHB = \angle C + \angle(\frac{B}{2})$ -------------------- (5)

$\angle AGH = \angle BAG + \angle(\frac{B}{2})$------------------(6)

(4), (5) &(6) $\longrightarrow$ $\angle AGH = \angle AHB$

Therefore, AG = AH Proved.

PRAISE THE LORD

- **DR. M. RAJACLIMAX**

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