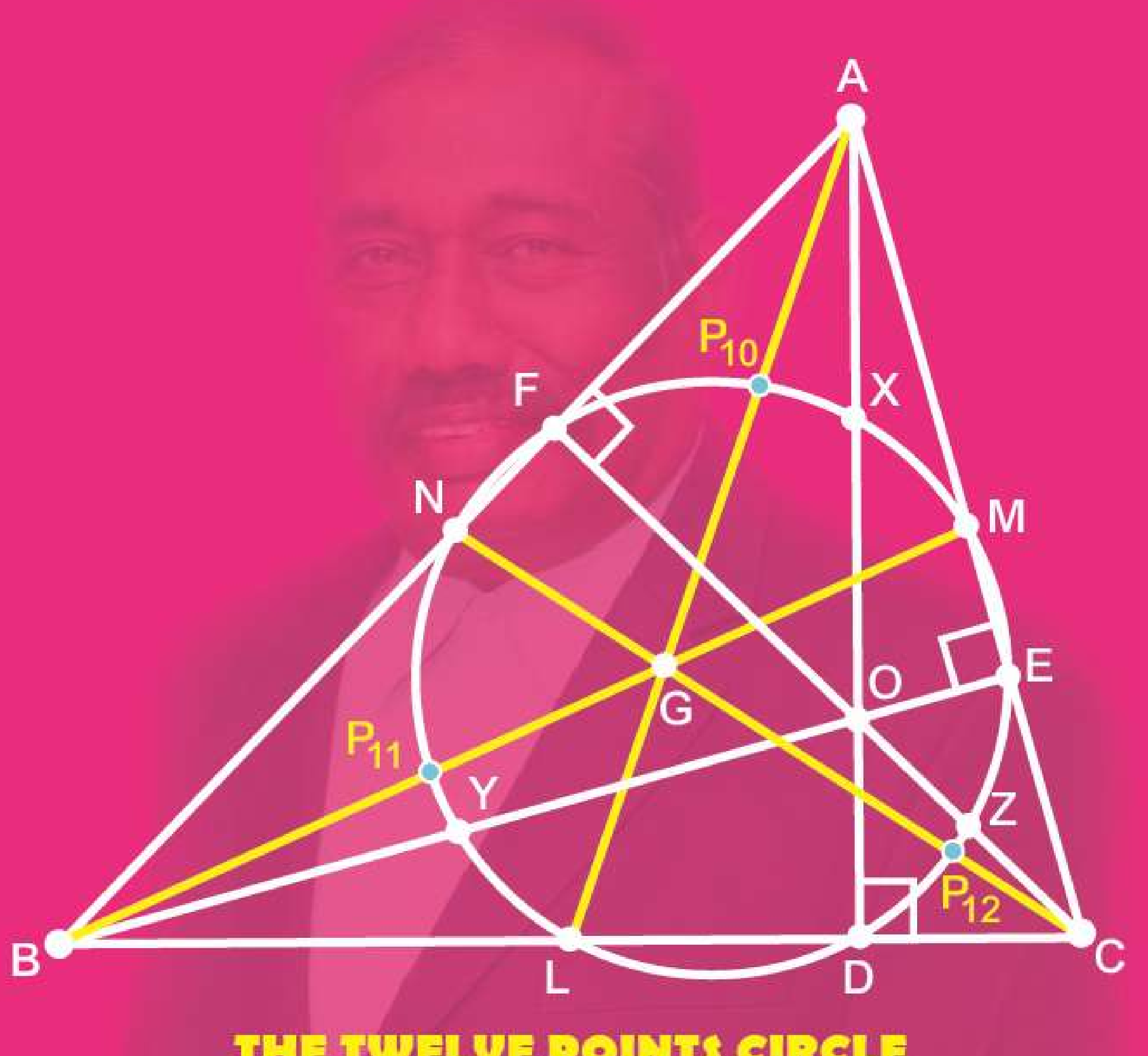
ADVANCED THEOREMS ON GEOMETRY

[NEW DISCOVERIES OF DR. M. RAJA CLIMAX]



THE TWELVE POINTS CIRCLE

G. Lakshmana Murthy, N. Sai Prasad Kumar
A. Madhumitha & P. Ponmalar Selvi

ADVANCED THEOREMS ON GEOMETRY

(NEW DISCOVERIES OF DR.M.RAJA CLIMAX, IRS)

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FOREWORD

Dr. Raja Climax's interest in intersection of lines, Ceva's and Menelaus' theorems related to Geometry of Concurrency, and Orthocenters, Medians and Chords of triangles is revealed explicitly in his present work.

The Concurrency Theorem developed by Dr. Raja Climax has far-reaching implications. This theorem provides a powerful tool for solving problems involving the intersection of lines and triangles. It provides a fundamental framework for analyzing complex Geometric configurations. As we continue to advance in fields like architecture, engineering, and computer science, the Geometry of concurrency will remain an essential tool for solving complex problems and creating innovative solutions.

Dr. Raja Climax has unearthed several new concepts on Orthocentre, Centroid, Altitudes, Angle Bisector and Medians. All his new theorems on these basic Geometric elements are pure novelties and innovations unknown and untold by any Mathematicians so far. These new theorems will pave way for numerous new future results, no doubt.

His new revelations about the existing Nine-Points Circle and upgrading the Nine-Points Circle into Twelve-Points Circle are amazing and praiseworthy. He has written a new history for the existing Nine-Points Circle Theorem. His Eight-Points Circle theorem is a new concept to study and cherish for the Geometry lovers. He has also made beautiful extensions to the concept of Simson line and brought out awesome discoveries which deserve to be called new theorems.

I honestly express my gratitude to Dr. Raja Climax for his contributions to Geometry and hope he'd continue to be in the world of Geometry for his own satisfaction.

Madurai **DR. B. MAYIL VAGANAN**
19.12.24

DR. S.R. SANTHANAM,
President
AIMER
(Association for International
Mathematics Education and Research)

PREFACE

The ancient branch of Mathematics, Geometry, dates back at least to the 5th Century BCE. It is still alive as young as before, because of its richness in the content and applications to various branches of knowledge. The Geometry of straight lines, polygons and circles is evergreen. Thousands of interesting results were found and still is being found also. It attracts not only mathematicians but also amateurs of all time.

Dr. Raja Climax is a genuinely interested and highly talented person in elementary geometry. Many think higher geometry concerns with projective geometry, non-euclidean geometry and differential geometry. But the scope in elementary geometry of lines, triangles and circles is enormous. One may think what new results can be produced in elementary geometry? But, the book authored by Dr. Raja Climax, makes us realize that the elementary geometry gives a lot of scope of future probing.

The deep interest in geometry by Raja Climax took him to present a nice paper on this topic in an International Conference in Bologna (Italy). This book will be certainly a feast to geometry lovers.

I wish good luck to Dr. Raja Climax for producing this book and many more such books to come in future.

Peddapuram, Andhra Pradesh **DR. S.R. SANTHANAM**
11.12.24

RAJA CLIMAX, THE GREATEST GEOMETER OF THE 21ST CENTURY

Among the branches of Mathematics, Geometry is believed to be the toughest. While so, a non-Mathematician mastering Geometry and discovering and formulating a host of new concepts and theorems in Geometry is wondrous and amazing. Yes, our Author, **Dr. M. Raja Climax, IRS** is not a Mathematician. He is just a graduate in Business Administration (BBA). He studied Mathematics only upto the Pre-University level (equivalent to the current 12th Standard). By profession also, he was only a tax officer and he is totally a stranger to the teaching or researching fields.

As close associates, we have been watching his role and performance in the arena of Geometry and we are astonished by the way he has been rocking the world of Geometry by discovering theorem after theorem. He has discovered more than 30 new theorems and framed more than 200 new challenging Geometric problems. He has been unearthing novel and innovative theorems so far unknown to the world of Geometry. In this book, we have edited the worthy and awesome discoveries he has made on Circles, Concurrency, Simson lines, Centroid, Orthocentre, Angle Bisectors, Altitudes, Cevians, etc. His contributions encompass all areas of Geometry. We acknowledge that he is the greatest Geometer of the 21st century. We wish that the Mathematicians of the world should take cognizance of his extraordinary prudence and give him the due and deserving honours.

Madurai **G. LAKSHMANA MURTHY, N. SAI PRASAD KUMAR,**
30.11.24 **A. MADHUMITHA & P. PONMALAR SELVI**

FROM THE GEOMETER

Until my age of 63, I remained detached from Geometrical researches. It was in 2016. I met **Mr. G. Lakshmana Murthy** of Warangal who stimulated my dormant interests in Geometry by volleying to me challenging problems and making me solve them. In this process, eventually, I started creating new corollaries by modifying the problems given by him. It was here, he encouraged me and shaped me from a solving Geometer into a research Geometer. I am glad and frank to admit that but for the practice and encouragement I was put through by him, I would not have risen to this level in Geometry.

In any field, sustained pursuit yields worthy results. Geometry is no exception. Relentless exploration in any area leads to new finds and the same thing happened with me in my Geometrical hunt. My prayers to the Lord Jesus and perseverance landed me in worthy discoveries, thanks to Him. A few days back, **Dr. S.R. Santhanam**, the Chief Functionary of AIMER (Association for International Mathematics Education and Research) suggested to me to bring out a book compiling all the Geometrical discoveries. Hence this book. I thank him not only for his suggestion but also for his valuable preface to this book. **Dr. B. Mayil Vaganan**, the Head of Department of Mathematics, Madurai Kamaraj University was kind enough to give his precious message appreciating this work. My special thanks to him.

I record my gratitude to **Mr. S. Kannan, IRS**, Ex. Principal Chief Commissioner of GST for his encouragement and support to me. I also thank Mr. N. Sai Prasad Kumar, Ms. A. Madhumitha and Ms. P. Ponmalar Selvi for editing this book to its present form.

Madurai **Dr. M. RAJA CLIMAX, IRS**
02.12.24

CONTENT

His (Dr. M. Raja Climax's) greatest work is his diving deeper and deeper into earlier Mathematicians' work and getting pearls out of it, like upgrading Nine-Points Circle into Twelve-Points Circle, finding the logic behind those additional three points, one more point on the Euler line (Fifth point), infinite number of Six-Points Circle, extension to Simson line and the list goes on. I wonder upon his approach of solving difficult problems created by others and equally creating challenging problems.

DR. B. MAYIL VAGANAN

Professor & HOD of Mathematics,
Member Syndicate & Member, Convenor Committee,
Madurai Kamaraj University, Madurai-625021.

I. THE TWELVE-POINTS CIRCLE

(PREVIOUSLY THE NINE-POINTS CIRCLE)

Theorem-1 : The Twelve-Points Circle Theorem

All Geometers know the Nine-Points Circle. The Nine-Points Circle was a special circle developed by **Karl Feuerbach** and **Orly Terquem** in the 19th century. They identified only the following nine points on the circle.

1. The feet of the three Altitudes -------- 3 points
2. The feet of the three Medians -------- 3 points
3. The midpoints of the line segment joining the vertex and the Orthocentre } ------- 3 points

Total --- 9 points

Now, **Dr. M. Raja Climax** has identified **three** more significant points and upgraded the circle to **Twelve-Points circle.** He has discovered the identity of the points intercepted by the circle in the upper portion of the three medians (so far undiscovered).

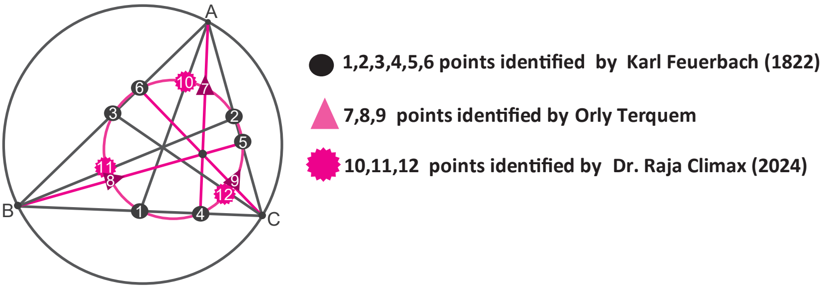

Figure : 1

The Twelve-Points Circle Theorem:

In the figure: 2, ΔABC is inscribed in the circle. The midpoints of its sides BC, CA & AB are L, M & N respectively. Its altitudes are AD, BE & CF and its Orthocentre is O. The feet of its altitudes are D, E & F. Besides, X, Y & Z are the midpoints of AO, BO & CO. The nine-points circle of the ΔABC goes through the nine points, D,E,F,L,M,N,X,Y&Z.

In the figure:2, we can see that the nine-points circle intersects each altitude at two points (D & X on altitude AD, E & Y on altitude BE and F & Z on altitude CF). Thus, for each altitude, two points are recognized as definite points of the Nine-Points Circle. Similarly, the Nine-Points Circle also intersects each median at two points (L & P_{10} on median AL, M & P_{11} on median BM and N & P_{12} on median CN).

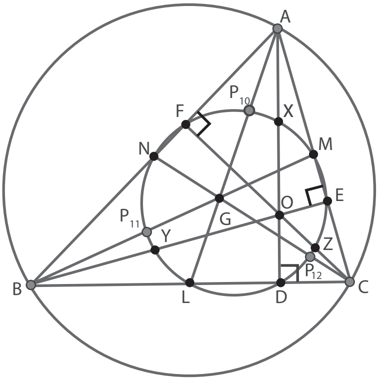

Figure : 2

Though, each median is intersected by this special circle at two points, only one is recognized so far by the Mathematicians and the other point is remaining unrecognized and ignored as an orphan point. For example, on the median AL, the point L is recognized as the foot of the median, but the point P_{10} on the same median is not recognized. Till date, the point P_{10} is left unidentified and unrecognized by the Mathematicians of the world. Similarly, the points P_{11} & P_{12} are also left ignored and unrecognized. Here, Raja Climax is proving that these points viz P_{10}, P_{11} & P_{12} are not orphan points but significant points. They are definite points. They have their own significance and property. They are cognizable points having their own uniqueness.

The significance and identity of the 10th, 11th and 12th points:

Now, let us see how Raja Climax establishes the identity and significance of the three points, P_{10}, P_{11} & P_{12}. In the figure:3, the median AL and the altitude AD are produced to meet their circumcircle at H & J respectively.

Now, we can call AH as the "Extended Median" or shortly as the "ex-median".

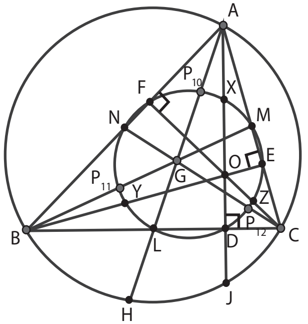

Figure : 3

Similarly, we can call AJ as the "Extended Altitude" or shortly as the "ex-altitude". Now, let us see how the ex-altitude and the ex-median help us understand the significance of the points on the altitudes viz X, Y & Z and the points on the medians viz P_{10}, P_{11} & P_{12}.

"In the figure, we can prove that XD = $\frac{1}{2}AJ$ = $\frac{1}{2}$ ex-altitude.

(using The Orthocentre Theorem). Now, we have clearly understood the significance of the point X from a different angle. ie It is not only that X is the midpoint of AO but also that XD is $\frac{1}{2}AJ$.

Now, let's see the significance of the point P_{10}. (Already we saw that XD is $\frac{1}{2}AJ$.)

Similarly, the distance between the point P_{10} and L is half ex-median AH. ie $LP_{10} = \frac{1}{2}AH = \frac{1}{2}$(ex-median)". (This is proved in the Lemma below.)

Lemma: The chord cut out by the Nine-Points Circle (LP_{10}) on a median is equal to half the extended median (AH).

In the figure: 4, ΔABC is inscribed in the circle. Its altitudes are AD & BE. O is its Orthocentre. Its median AL is produced to meet the circumcircle at H. Its Nine-Points Circle passes through L, D & E and cuts AO at X and AL at P_{10}.

Prove that $LP_{10} = \frac{1}{2}AH$.

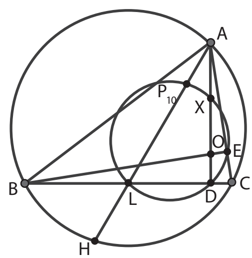

Figure : 4

Construction :

Join LX (See figure: 5). Mark the circumcentre of the bigger circle P. Join OP. Let OP & LX cut at Q. Draw the diameter (of the bigger circle) APR from A. Join RH and X P_{10}

Proof:

Since LX subtends an angle of 90° at D in the Nine-Points circle, LX is the diameter of the Nine-Points circle.

OQP is the Euler's line and Q is the midpoint of OP and also is the circumcentre of the Nine-Points Circle.

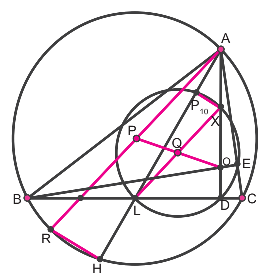

Figure : 5

$\therefore$ LX = $\frac{1}{2}AR$ ------------ (1) [The diameter of the Nine-Points Circle is half the diameter of the circumcircle]

We know X is the midpoint of AO and Q is the midpoint of PO.

$\therefore$ XQ ∥ $AP \Rightarrow$ XL ∥ AR

$\therefore \angle RAL = \angle XLA$ [alternate angles] ---------- (2)

Now, in Δ P_{10}LX & ΔHAR

∠RHA = $\angle LP_{10}X = 90°$ [Angles subtended by diameters]

$\angle XLP_{10} = \angle RAH$ [(2) above]

$\therefore \Delta\ RAH \sim \Delta XLP_{10} \Longrightarrow \frac{RA}{XL} = \frac{AH}{LP_{10}} = \frac{RH}{XP_{10}}$ --------- (3)

From (1), $\frac{RA}{XL} = 2$ --------------------------------------- (4)

$\therefore$ LP_{10} = $\frac{1}{2}AH$ ------------------------------ Proved.

It's now clear that P_{10} is not a random point or orphan point but a cognizable point with its own relationship with the ex-median. We can also similarly show that the points P_{11} & P_{12} are also not orphan points but cognizable points having their own significance. ie MP_{11} and NP_{12} are equal to half of their respective ex-medians.

ie $MP_{11} = \frac{1}{2}\ BK$

& $NP_{12} = \frac{1}{2}\ CI$

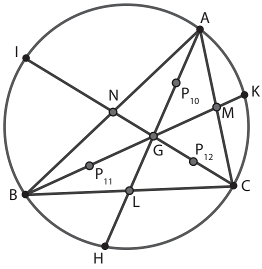

Figure : 6

Another method to identify the 10th, 11th & 12th points:

Raja Climax has also developed another method to establish the identity and significance of the 10th, 11th & 12th points, ie P_{10}, P_{11} & P_{12}.

We saw above that L$P_{10} = \frac{1}{2}AH$, (see figure:7)

In figure: 7 , AH is the extended median, G is the Centroid and P_{10} is the point at which the special circle (Nine Points circle) intersects the median AL in its upper portion. Now,

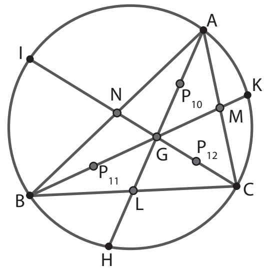

Figure : 7

$\frac{AG}{GL} = 2$ & $\frac{AH}{LP_{10}} = 2$ (Proved in the above Lemma)

$\Longrightarrow \frac{AG}{GL} = \frac{AH}{LP_{10}} = \frac{(AH-AG)}{(LP_{10}-GL)} = 2$ (*Componendo and dividendo*)

$\Longrightarrow \frac{GH}{GP_{10}} = 2$

ie The Centroid G divides the segment HP_{10} in the ratio of 2:1.
$\Longrightarrow$ P_{10} is a point on the extended median AH such that GH = 2GP_{10}
Similarly, we can prove that GK = 2GP_{11} $\& \ GI = 2GP_{12}$
Now, the identity of the points $P_{10}, P_{11} \& P_{12}$ has been established in another method using the Centroid.

Conclusion:
We have now 12 definite points of the ΔABC on this special circle as detailed below:

1. The feet of the three Altitudes ------ 3 points
2. The feet of the three Medians ----- 3 points
3. The midpoints of the line joining the vertex and the Orthocentre } ----- 3 points
4. The points on the medians at a distance of half ex-median from the foot (*or the points in the upper portion of the extended medians at a distance of half the line segment joining the Centroid and the circle*) } ----- 3 points

Total ---- 12 points

Let us therefore, call this special circle of the ΔABC as the **"Twelve-Points Circle"** and hereafter let us call this special circle's theorem as the **"Twelve-Points Circle Theorem"**.

*****

II. CENTROID AND THE TWELVE-POINTS CIRCLE

(The Unknown Properties of Centroid)

The Centroid is the meeting point (concurrent point) of the three medians of a triangle. So far, all that is known about the Centroid is that it divides the median in the ratio of 2:1. But now, apart from this vital property, **Raja Climax** has discovered certain novel properties of Centroid in relation to the Twelve-Points circle (TPC) of a triangle and the same are discussed here as 'The CENTROID-TPC THEOREMs'.

Theorem 2: The Centroid - TPC Theorems

The Centroid - TPC Theorem I

Every triangle has a Twelve-Points circle (TPC). This Twelve-Points circle is formed by the twelve significant points located on the three altitudes and the three medians of the triangle. The altitudes and the medians (six in number) contribute two points each to this special circle. The three altitudes meet at the Orthocentre and the three medians meet at the Centroid. Therefore, in the study of the Twelve-Points circle, the Orthocentre and the Centroid assume significance.

As for the Orthocentre, a relationship has already been established by the Geometers with reference to the Twelve-Points circle. ie, Any straight line drawn from the Orthocentre to the circumference is bisected by the Twelve-Points circle. (See the picture below).

In the adjacent picture, ΔABC is inscribed in its circumcircle. AL, BM & CN are its altitudes and H is its Orthocentre. Its Twelve-Point circle is depicted in red colour. HJ & HK are lines shot from H, the Orthocentre, crossing the Twelve-Points circle at P & Q respectively. Now, HP = PJ and HQ = QK.

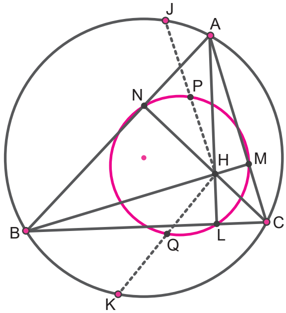

Figure : 8

This is an already-known relationship between the Orthocentre and the Twelve-Points circle. Now, the question arising is whether any such relationship is there between the Centroid and the Twelve-Points circle. So far, the Geometers could not establish the existence of any remarkable relationship. But now, **Raja Climax** has found out some beautiful relationships between the Centroid and the Twelve-Points circle. The first one is 'The Centroid-TPC Theorem I' which is discussed below.

'The Centroid- TPC Theorem I' in textual terms:

The intercepts (chords) cut out by the circumcircle and the Twelve-Points Circle on any line drawn through the Centroid of a triangle will be in the ratio of 2:1. (ie MN = 2XY, see the picture)

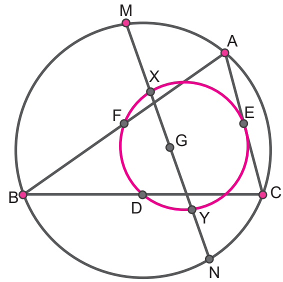

Figure : 9

'The Centroid - TPC Theorem I' in Geometric terms:

In the picture, ΔABC is inscribed in the circle. D, E & F are the midpoints of BC, AC & AB respectively and G is the Centroid. The Twelve-Points circle (TPC) of the ΔABC is drawn through D, E & F. A random line drawn through G intersects the circumcircle at M & N and the TPC at X&Y as shown in the picture. Now, the Centroid-TPC Theorem I says, **MN = 2 XY**.

Proof :

Construction:

Mark the circumcentre O and the centre of the Twelve Points Circle H (See picture). Draw OJ ⊥ MN and HK ⊥ MN. Join OM, HY, OG & GH.

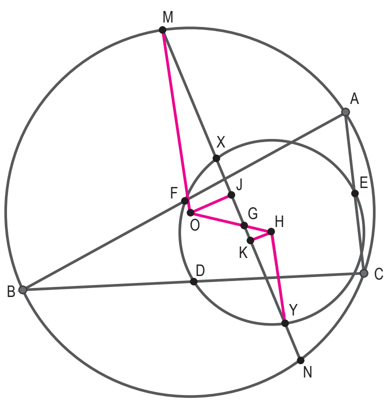

Figure : 10

OGH is part of Euler line.

⇒ OGH is a straight line.

$\Rightarrow\ \Delta OGJ \sim \Delta HGK$

we know, $\frac{OG}{GH} = 2$ (Circumcentre, Centroid & TPC centre)

$\Rightarrow \frac{OJ}{HK} = 2$ --------------------- (1)

In ΔOJM & ΔHKY,

$\Rightarrow \frac{OM}{HY} = 2$ (Radii of circumcircle & TPC) -------------- (2)

Both are right Δs ------------------------- (3)

(1), (2) & (3)→

$\Delta OJM \sim \Delta HKY$ (Side & hypotenuse principle)

$\Rightarrow \frac{JM}{KY} = 2$

$\Rightarrow \frac{\frac{1}{2}(MN)}{\frac{1}{2}(XY)} = 2$

$\Rightarrow MN = 2XY$ -------------------------Proved.

*****

The Centroid-TPC Theorem II

The Centroid-TPC Theorem II in Textual Terms:

When a random line is drawn through the Centroid of a triangle intersecting the circumcircle and the TPC (Twelve-Points Circle), the line segment joining the point of intersection of the circumcircle and that of the TPC is divided by the Centroid in the ratio of 2:1. (Let this be given clarity by a statement in Geometric terms. See below.)

The Centroid-TPC Theorem II in Geometric Terms:

In the picture, ΔABC is inscribed in the circle. D, E & F are the midpoints of BC, AC & AB respectively and G is the Centroid. The Twelve-Points circle (TPC) of the ΔABC is drawn through D, E & F. A random line drawn through G intersects the

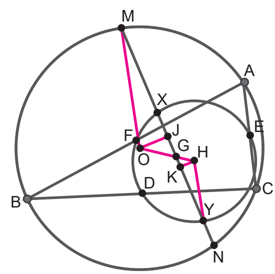

Figure : 11

circumcircle at M & N and the TPC at X & Y as shown in the picture.

The Centroid-TPC Theorem II says: MG:GY = 2:1 or MG = 2GY. **(ie G is also the Centroid for the triangle with M as a vertex and Y as the midpoint of its opposite side.)**

Proof:

Construction: Mark the circumcentre O and the centre of the Twelve-Points Circle H inside the picture. Draw OJ $\perp MN$ and HK $\perp$ MN. Join OM, HY, OG & GH.

In the previous theorem, we proved,

$\Delta OJM \sim \Delta HKY$

$\Rightarrow \angle OMG = \angle HYG$

Also we have, $\angle OGM = \angle HGY$

$\Rightarrow \Delta OMG \sim \Delta HYG$

But we know, $\frac{OG}{GH} = 2$

$\Rightarrow \frac{MG}{GY} = 2$

MG = 2GY ----------- Proved

[ie G is also the Centroid for the triangle with M as a vertex and Y as the midpoint of the opposite side. (See the picture and the explanation below.)]

In the previous picture, join OY and draw a perpendicular line to OY through Y to intersect the circumcircle at R & S. Join MR & MS. Now, G is the Centroid for ΔMRS also.

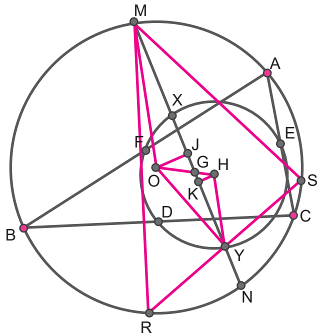

Figure : 12

<u>Corollary</u>:

Since O is the circumcentre of ΔMRS & G has been proved as its Centroid, it follows that H is the centre of its Twelve-Points circle. Hence, the smaller circle is also the TPC for ΔMRS.

*****

The Centroid-TPC Theorem III

The Centroid-TPC Theorem III is almost a corollary of the Centroid-TPC Theorem II. The theorem is stated in textual and Geometric forms below:

The Centroid-TPC Theorem III in textual terms:

The Centroid of a triangle not only divides its Median in the ratio of 2:1 but also the portion of its 'Extended Median' segmented by its upper intersection point of the Twelve-Points circle and its (the Extended Median's) end point with the circumcircle in the same ratio ie 2:1.

The Centroid-TPC Theorem III in Geometrical terms:

In the picture, ΔABC is inscribed in its circumcircle. D, E & F are the midpoints of its sides BC, AC & AB respectively. The Twelve-Points circle of ΔABC is drawn through D, E & F. The median AD is extended to meet the circumcircle at P and AP is the Extended Median and G is the Centroid. AP intersects the Twelve-Points circle at X. In this scenario, already we know that AG:GD = 2:1. Now, **the Centroid-TPC Theorem III says, PG:GX = 2:1.**

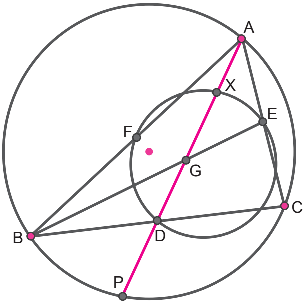

Figure : 13

Proof:

We know, $\frac{AG}{GD}$ = 2 ----------------(1)

From the Centroid-TPC theorem I (proved above),

we know, $\frac{AP}{XD}$ = 2 -----------------(2)

(1)&(2)—› $\frac{(AP-AG)}{(XD-GD)}$ =2 (componendo dividendo)

$\Longrightarrow$ $\frac{PG}{GX}$ **= 2 -------------------(proved)**

*****

The Centroid-TPC Theorem IV

The above proof of the Centroid-TPC Theorem III can also be used to prove that any line passing through the Centroid will get divided in the ratio of 2:1 in its segment joining its intersection with the two circles.

The Centroid-TPC Theorem IV in Geometric terms:

In the picture, ΔABC is inscribed in its circumcircle. D, E & F are the midpoints of its sides BC, AC & AB respectively. The Twelve-Points circle of ΔABC is drawn through D, E & F. PQRS is any random straight line drawn through G as shown in the picture. Now, the Centroid-TPC Theorem IV says, SG:GQ = 2:1.

Proof:

PQRS is a straight line passing through centroid G.

As per the Centroid-TPC Theorem I,

$$\frac{PS}{QR} = 2 \qquad \text{----------------------------- (1)}$$

As per the Centroid-TPC Theorem II,

$$\frac{PG}{GR} = 2 \qquad \text{---------------------------- (2)}$$

(1) & (2) →

$\frac{\mathrm{PS-PG}}{\mathrm{QR-GR}}$ = 2 (Componendo Dividendo)

⇒ $\frac{SG}{GQ}$ = 2 ---------------------(Proved)

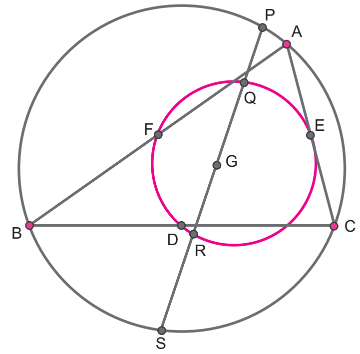

Figure : 14

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The Centroid-TPC Theorem V

The Centroid-TPC Theorem V in textual terms:

When any two random straight lines are drawn through the Centroid of a triangle intersecting the circumcircle and the TPC, two sets of 'four points' are created. The Centroid-TPC Theorem V says each set of 'four points' represents a cyclic quadrilateral. (Let this be given clarity by a Geometrical version.)

The Centroid-TPC Theorem V in Geometrical terms:

In the picture, ΔABC is inscribed in a circumcircle. D, E & F are the midpoints of the sides BC, AC & AB respectively. G is its Centroid. Its TPC is drawn through D, E & F. Two lines, KN & PS are drawn through G intersecting the two circles at K, L, M &N and P, Q, R & S respectively as shown in the picture. The Centroid-TPC Theorem V says RMNS and PQLK are concyclic.

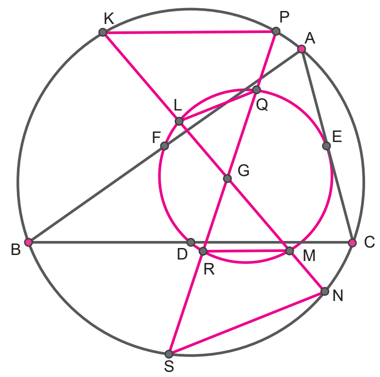

Figure : 15

Proof:

The lines PQRS and KLMN pass through G, the Centroid.

As per the Centroid-TPC Theorem IV,

$\frac{NG}{GL} = 2$ & $\frac{SG}{GQ} = 2$

And $\angle LGQ = \angle SGN$

$\Rightarrow \Delta SGN \sim \Delta QGL$ (SAS principle)

$\Rightarrow \angle LQG = \angle NSG = \angle NSP$ ------ (1)

$\angle NSP = \angle NKP$ (same segment) ---- (2)

(1) & (2) $\rightarrow \angle LQG = \angle NKP = \angle LKP$

$\Rightarrow$ LKPQ is concyclic.

Similarly, we can prove that RMNS is concyclic

------------- Proved.

Corollary: 1

LQ || SN & SN = 2 LQ and KP || RM & KP = 2RM

Corollary: 2

In the picture, ΔABC is inscribed in a circumcircle. D, E & F are the midpoints of the sides BC, AC & AB respectively.

G is its Centroid. Its TPC is drawn through D, E & F. AP, BQ & CR are the extended medians meeting the TPC at X, Y & Z respectively. Now, the following are concyclic: **AXEQ, QEZC, CZDP, PDYB, BYFR, RFXA, RFEQ, AXZC, QEDP, CZYB, PDFR & BYXA.**

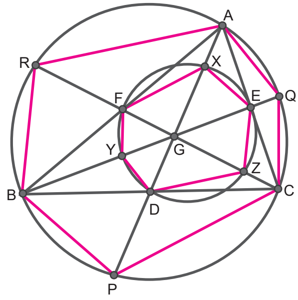

Figure : 16

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The Centroid-TPC Theorem VI

The Centroid-TPC Theorem VI in textual terms:

When a straight line is shot from the Centroid of a triangle to its circumcircle intersecting its TPC, the product of the line segments it makes with the circumcircle and with the TPC is constant.

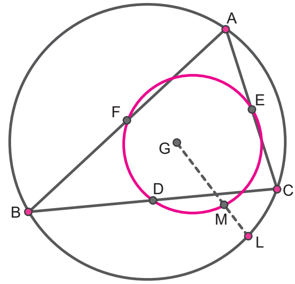


Figure : 17

The Centroid-TPC Theorem VI in Geometrical terms:

In the picture, ΔABC is inscribed in a circumcircle. D, E & F are the midpoints of the sides BC, AC & AB respectively. G is its Centroid. Its TPC is drawn through D, E & F. L is any point on the circumference of the circumcircle. GL intersects the TPC at L. Now, The Centroid Theorem VI says, **(GL × GM)** is a constant for all lines shot from G.

Proof:

Construction:

Mark another random point K on the circumcircle and join GK to intersect the TPC at N.

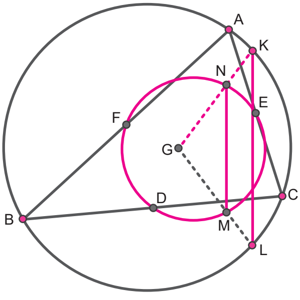


Figure : 18

To Prove:

GM×GL = GN×GK = constant for all lines shot from G.

Proof:

From the Centroid-TPC Theorem V and its corollary 2, we know that KLMN is a cyclic quadrilateral.

$\Rightarrow$ $\boldsymbol{GM \times GL = GN \times GK}$ (Chords KN & LM are intersecting at G outside their circle.)

This equality can be proved in a similar way for any line shot from G to the circumference. - Proved.

*****

III. THE NEW CONCEPT OF EIGHT-POINTS CIRCLE

Theorem-3 : The Eight-Points Circle Theorem

We all know that every triangle has a Nine-Points Circle (Discussed as the Twelve-Points Circle in the previous chapter). Now, it has come to light that in addition to this Nine-Points Circle, every triangle also has an Eight-Points Circle. This Eight-Points Circle is a new special circle discovered by Dr. M. Raja Climax. In the figure: 19, the red circle is the Eight-Points Circle. The points G, O, X, Y, Z, S, T & U are the eight significant points lying on this circle. GO (the line segment joining the Centroid G and the Orthocentre O) is the diameter of this circle.

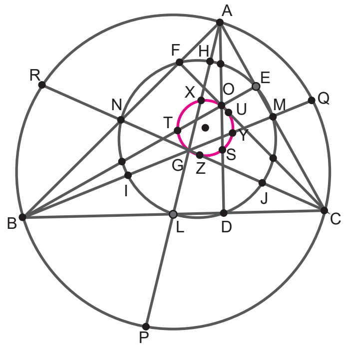


Figure : 19

In the figure: 19, the Twelve Points Circle is portrayed in black colour and the Eight-Points Circle is shown in red colour. This Eight-Points Circle passes through the following eight points.(see the above picture)

1. The Orthocentre of the ΔABC – O (one point)
2. The Centroid of the ΔABC – G (one point)
3. Three points on the medians viz X, Y & Z

   These points are on the medians such that XL = LP, YM = MQ and ZN = NR. [Also HA = HX; IB = IY; and JC = JZ. Already, it has been proved in the Twelve-Points Circle Theorem that HL = $\frac{1}{2}$ AP, IM = $\frac{1}{2}$ BQ and JN = $\frac{1}{2}$ CR.]
4. Three points on the altitudes viz S, T & U

   These points divide their respective altitudes in the ratio 2:1.

   ie AS:SD = BT:TE = CU:UF = 2:1.

TOTAL = 8 points. These eight points viz O, G, X, Y, Z, S, T & U form this Eight-Points Circle.

It is to be noted that the diameter of this Eight-Points Circle is the line segment joining the Orthocentre and Centroid ie OG and the midpoint of OG is the circumcentre of this Eight Points Circle. Also, OG is part of the Euler line.

Before setting to prove the Eight-Points Circle Theorem, let us prove the following lemma which will be useful in proving the Eight-Points Circle Theorem.

Lemma :

ΔABC is inscribed in the circle. AD is its altitude and O is its Orthocentre. Its median AL is produced to meet the circle at P. X is a point on AL such that LX = LP. K & H are the midpoints of AO & AX respectively. Prove that KH ⊥ AL and OX ⊥ AL.

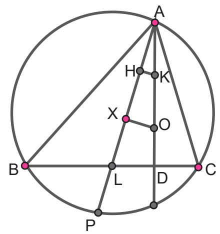


Figure : 20

Proof :

Join KL.

LX+XH = $\frac{1}{2}XP + \frac{1}{2}AX = \frac{1}{2}AP = \frac{1}{2}$ extended median.

∴The Nine-Points Circle (also called the Twelve-Points Circle) of ΔABC will pass through the points D, K ,H & L and LK will be its diameter.

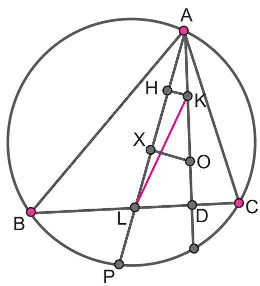


Figure : 21

$\therefore \angle KHL = 90°$

$\therefore KH \perp AL$-------------------(1)

In ΔAXO, K & H are the midpoints of AO & AX respectively.

$\therefore KH \parallel XO$ ------------------ (2)

(1) & (2) ⟶

OX $\perp AL$ --------------------------------Proved

From (1), $KH \perp AL$-------------------Proved

*****

In the figure: 22, AD, BE & CF are the altitudes of Δ ABC and O is its Orthocentre. The medians AL, BM & CN are produced to meet the circumcircle at P, Q & R respectively and G is the Centroid. X is a point on AP such that LP = LX; Y is a point on BQ such that MQ=MY; and Z is a point on CR such that NR = NZ.

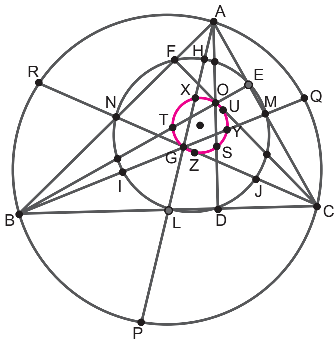


Figure : 22

The points S, T & U divides their respective altitudes AD, BE & CF in the ratio 2:1.

Prove : GZSYUOXT is concyclic ie to prove that the eight points viz X, Y, Z, S, T, U, the Orthocentre O and the Centroid G are concyclic and that OG is the diameter of that circle.

Proof:

In the above Lemma, we proved **OX** $\perp$ AL. Similarly, we can prove

OY $\perp BM$ & OZ $\perp CN$.

Hence, $\Delta\, OXG, \Delta OYG$ & ΔOZG are right triangles with OG as their common hypotenuse.

Hence, the points X ,Y, Z, O & G are concyclic and the midpoint of OG (hypotenuse) is the centre of that circle ----- (1)

It is given that the point S divides AD in the ratio 2:1

ie $\frac{AS}{SD} = \frac{2}{1}$ -- (2)

Also given that AL is a median and G is the Centroid.

$\Rightarrow$ $\frac{AG}{GL} = \frac{2}{1}$ -- (3)

From (2)&(3), $\frac{AG}{GL} = \frac{AS}{SD}$

$\Rightarrow$ GS ∥ LD

$\Rightarrow\ \angle ASG = \angle ADL = 90°$ (∵ AD is altitude)

⇒ OSG is a right Δ with OG as hypotenuse. ------ (4)

Similarly,

we can prove that ΔOTG, & ΔOUG are also right triangles on the common hypotenuse OG. ---- (5)

(4) & (5) ⟶ O,U,S,G &T are concyclic -------------- (6)

(1) & (6) ⟶ G,Z,S,Y,U,O,X&T are concyclic

The 8 points G,Z,S,Y,U,O,X&T are concyclic and the diameter of the circle is OG and the circumcentre is the midpoint of OG.

It is to be noted that the diameter of this Eight-Points Circle is the line segment joining the Orthocentre and Centroid ie OG. The midpoint of OG is the circumcentre. Also, OG is part of the Euler line.

Corollary :

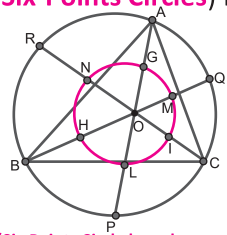
The circumcentre of the Eight-Points Circle is the 5th point on the Euler's line. (Already the mathematicians have recognized four points on the Euler line, viz the circumcenter of ΔABC, the Orthocentre of ΔABC, the Centroid of the ΔABC and the centre of the Twelve-Points Circle of the ΔABC.)

*****

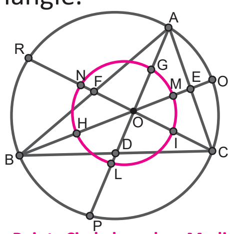
IV. THE INFINITE NUMBER OF SIX-POINTS CIRCLES

Theorem-4 : The Infinite Number of Six-Points Circles Theorem

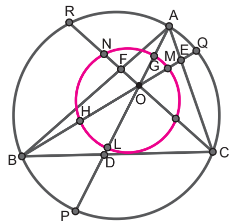
There was only one special circle for a triangle viz the Nine Points Circle (now, upgraded by Raja Climax as the Twelve-Points Circle). Now Raja Climax has discovered that there are infinite number of special circles (**Six-Points Circles**) for a triangle.



(Six Points Circle based on Angle Bisectors)



(Six Points Circle based on Medians)



(Six Points Circle based on Random concurrent cevians)

All these Six-Points circles also have the same special features of the Nine-Points Circle (Twelve-Points Circle).

New Six-Points Circles based on angle bisectors and medians:

In the figure: 23, AP, BQ & CR are the angle bisectors of $\angle A, \angle B \& \angle C$ (of ΔABC) respectively and they are concurrent at O the incentre. Now, there are six line segments viz, OA, OB, OC, OP, OQ & OR and their midpoints are G,H,I,L,M & N respectively.

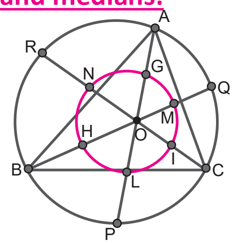


Figure : 23

(Based on Angle Bisectors)

When we check all these six points for concyclicness, it is surprising that all these six points are concyclic and we got a beautiful Six-Points circle.

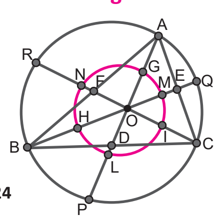


Figure : 24

(Based on Medians)

When the same process is repeated for the three medians of the triangle, again, we get a beautiful Six-Points Circle.

It is very clear that every triangle has a Six-Points Circle based on their medians and another Six-Points Circle based on their angle bisectors.

A new understanding of the Nine-Points Circle:

In the figure: 25 , the altitudes AD, BE & CF are extended to meet the other side of the circle at P, Q & R respectively.

Now, D is the midpoint of OP; E is the midpoint of OQ and F is the midpoint of OR. Also, G is the midpoint of AO; H is the midpoint of BO and I is the midpoint of CO.

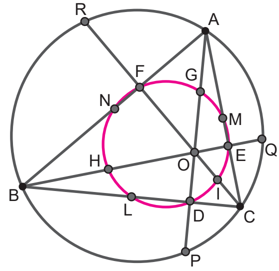


Figure : 25

There are totally six midpoints for the six line-segments, OA, OB, OC, OP, OQ & OR. All these six midpoints are concyclic in the Nine-Points Circle. What is to be noted here is that there are three altitudes (AD, BE & CF) which are concurrent at O, the Orthocentre. These three altitudes meet the circumference at P, Q & R respectively. And these three extended altitudes are split into six line segments viz OA, OB, OC, OP, OQ & OR and when their midpoints are marked, all those midpoints are concyclic in a Six-Points Circle. The Nine-Points Circle is therefore, a Six-Points Circle formed based on the altitudes of the triangle and co-incidentally, it also goes through the feet of the medians.

New Six-Points Circle based on random concurrent cevians:

Now, let us take at random, any three concurrent cevians and repeat the above process and test whether there is a Six-Points Circle for the midpoints of the six line-segments created by those three concurrent cevians.

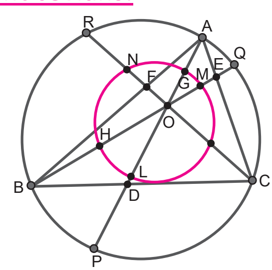


Figure : 26

(Based on Random Concurrency)

In the figure:26, AD, BE & CF are three random cevians of ΔABC concurrent at O and AP, BQ & CR are the extended cevians.

Now, there are six line segments viz, OA, OB, OC, OP, OQ & OR created by those cevians and their midpoints are G, H, I, L, M & N respectively. When we check all these six points for concyclicness, it is surprising that all these six points are concyclic and we got a beautiful Six-Points Circle. So, we have understood that every triangle has infinite number of Six-Points Circles.

All Six-Points Circles also have the same special features of the Twelve-Points Circle:

All the above Six-Points Circles have the same special features of the Nine-Points Circle as explained below:

1. The radius of all these 'Six-Points Circles' of Δ ABC is half the radius of its circumradius.

2. Any straight line drawn from the concurrent point O to the circumcircle of Δ ABC is bisected by the circumference of the 'Six-Points Circle'.

   In the figure:27, the line OJ is bisected at X by the circumference of the Six-Points Circle. Similarly, the line OK is bisected at Y by the circumference of the Six-Points Circle.

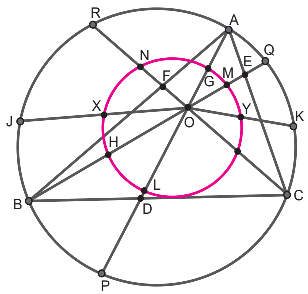


Figure : 27

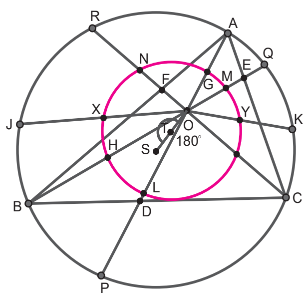


Figure : 28

3. In addition to the above two special features, one more new feature has also come to light. ie, the circumcentre of the triangle, the centre of the Six-Points Circle and the point of concurrency (O) are collinear. It may be noted that in the case of the Nine-Points Circle, the Orthocentre (the point of concurrency), the circumcenter of the Nine-Points Circle and

the circumcenter of the circumcircle of the triangle are collinear.

All the above features of the Six-Points Circle can be easily proved Geometrically.

Significance:

So far Mathematicians are believing that there is only one special circle for a triangle and that is the Nine-Points circle. But Raja Climax has proved that there are infinite number of Six-Point Circles for a triangle with the same special features of the Nine-Points Circle viz,

1. **Any line drawn from the point of concurrency of the Six-Points Circle, when produced to the circumcircle, will be bisected by the Six-Points Circle.**
2. **The diameter of all such Six-Points Circle is half the diameter of the circumcircle.**

V. ADVANCEMENTS ABOUT SIMSON LINE

Theorem 5: The infinite number of Simson lines theorem

Introduction:

According to the concept of Simson line, for a ΔABC, only one Simson line can be drawn to a triangle with reference to a point P on the circumference. This was done by dropping perpendiculars from point P to the three sides. **Raja Climax** has proved that Simson lines can be drawn not only by dropping perpendiculars but also by dropping lines with any angles, say, 30°, 37°, 38.15° and so on. In this way, infinite number of Simson lines can be drawn with reference to a point P on the circumference. This new concept is discussed here in the name of **"Infinite number of Simson lines Theorem".**

The Theorem in Geometric terms:

Δ ABC is inscribed in its circumcircle. P is a point on the minor arc AB. PL, PM and PN are straight lines drawn from P to CB (produced), AB and AC respectively such that $\angle PLC = \angle PMA = \angle PNA = \theta$. Prove that L, M & N are collinear.

To Prove:

L, M & N are collinear.

Construction :

Join PA & PB

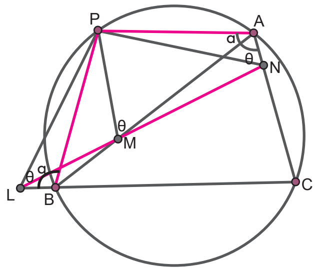


Figure : 29

Proof :

AMB is a straight line

$\Longrightarrow \angle AML + \angle LMB = 180°$ ----------------- (1)

To prove that LMN is a straight line, it is enough to prove that

$\angle AML + \angle AMN = 180°$ ---------------------- (2)

From (1) & (2), it is enough to prove

$\angle LMB = \angle AMN$ ------------------------------ (3)

Now, $\angle ANP = \angle AMP = \theta$ (given)

$\Longrightarrow$ APMN is concyclic.

$\Longrightarrow \angle APN = \angle AMN$ -------------------------- (4)

$\angle PLB = \angle PMA = \theta$ (given)

ie, Exterior angle is equal to interior opposite angle and hence PLBM is concyclic.

$\Longrightarrow \angle LMB = \angle LPB$ --------------------------- (5)

APBC is concyclic.

$\therefore \angle PAC = \angle PBL$ (Exterior angle and interior opposite angle)

$ie\ \angle PAN = \angle PBL$ ---------------------- ------ (6)

$\therefore$ In ΔPAN & ΔPBL, the third angles are equal.

ie. $\angle LPB = \angle APN$ ---------------------------- (7)

(4), (5) & (7) $\longrightarrow$

$\angle LMB = \angle AMN$ -------------------------------(8)

(3) & (8) $\longrightarrow$ LMN is a straight line (Simson's line)

------------------ **Proved**

Like LMN, we can generate any number of Simson lines by changing the value of θ. Hence this theorem is called the Infinite Number of Simson Lines Theorem.

*****

Theorem 6 : Length of the Simson line segment

So far, there is no formula or method to measure the length of the Simson line segment. **Raja Climax** has discovered a formula for measuring the Simson line segment. In the figure:30, ΔABC is inscribed in the circle. The Simson line segment with respect to the point P on the minor arc AB is LMN.

The measurement of this line segment:

LN = PC $\times$ SinC

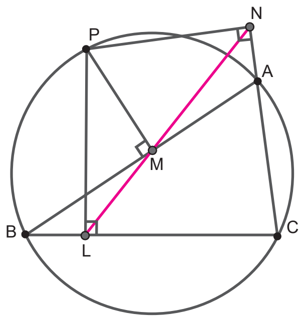


Figure : 30

Given:

ΔABC is inscribed in a circle. P is a point on the minor arc AB.

PL, PM & PN are the perpendiculars drawn to BC, AB & AC respectively and LMN is the Simson line segment.

To Prove: LN = PC × SinC

Construction:

Mark the centre of the circle 'O' and draw PT, the diameter through O.

Join PA, PB & TC.

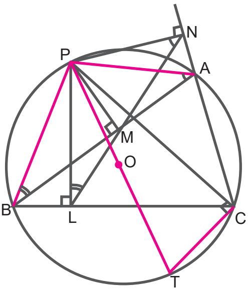


Figure : 31

Proof:

$\angle PMB = \angle PLB = 90°$ (given)

$\Longrightarrow$ PBLM is concyclic

$\Longrightarrow \angle PBM = \angle PLM$ -----------------------(1)

$\angle PMA + \angle PNA = 90° + 90° = 180°$

$\Longrightarrow PMAN$ is concyclic

$\Longrightarrow \angle PNM = \angle PAM$ ----------------------(2)

$\Longrightarrow (1)$ & $(2) \rightarrow \Delta PAB \sim \Delta PNL$

$\Longrightarrow \frac{PA}{PN} = \frac{AB}{LN}$ ------------------------ (3)

In ΔPTC & ΔPAN,

$\angle PTC = \angle PAN$ $[\because Exterior\ angle = Interior\ opp.angle]$

$\angle PCT = \angle PNA = 90°$ [∠PCT borne by diameter]

$\therefore \Delta PTC \sim \Delta PAN$

$\Longrightarrow \frac{PT}{PA} = \frac{PC}{PN}$ --------------------------- (4)

(3) & (4) → $LN = \frac{PC \times AB}{PT}$ ----------------------- (5)

But from the figure: 20

AB = 2R SinC ------------------------------ (6)

& PT = diameter = 2R Sin90° ----------------- (7)

(5), (6) & (7) →

LN = PC × SinC ---------------------------- Proved

<u>The Utility of the result:</u>

This result measures the Simson line segment LMN. ie.

$$LMN = \frac{PC \times AB}{D} = \frac{PC \times AB}{2R} = \boldsymbol{PC \times Sin\ C}$$

$[\because AB = 2R\ Sin\ C]$

In the figure:32, ΔABC is inscribed in the circle. P, Q & R are points on the minor arcs AB, AC and BC respectively. LMN, DEF & XYZ are the Simson lines drawn with respect to the points P, Q & R.

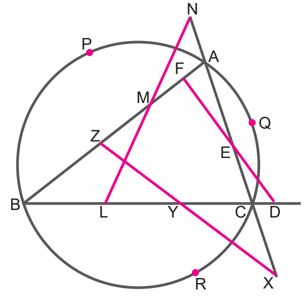


Figure : 32

Now, $\boldsymbol{LN = PC \times Sin\ C}$

$\boldsymbol{DF = QB \times Sin\ B}$

$\boldsymbol{XZ = RA \times Sin\ A}$

*****

Theorem 7 : Length of the Simson line segment with reference to angles other than 90°

Raja Climax has also framed a formula for measuring a Simson line segment with reference to a point P, when a Simson line is created by dropping lines making an angle of $\theta°$ (not by making 90°).

In the figure:33, Δ ABC is inscribed in the circle. P is a point on the minor Arc AB. PX, PY & PZ are lines dropped to BC (extended), AB & AC respectively such that $\angle PXC = \angle PYA = \angle PZA = \theta$.

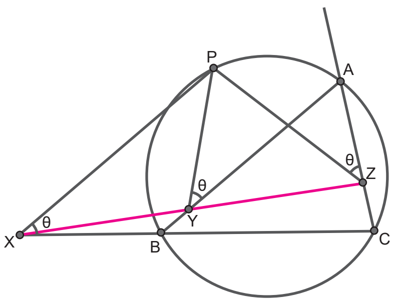


Figure : 33

Now XYZ is a straight line. Here, the length of the line XZ is given by the following formula. XZ= $\boldsymbol{\frac{PC \times SinC}{Sin\theta}}$

When $\theta = \angle C, XZ = PC$.

Now let us see how this is proved.

In the figure:34, ΔABC is inscribed in a circle. P is a point on the minor arc AB. PL, PM & PN are perpendiculars drawn from P to BC, AB & CA (produced) respectively. LMN is the Simson line (for 90°). PX, PY & PZ are straight lines drawn to CB (produced), AB & AC respectively such that $\angle PXC = \angle PYA = \angle PZA = \theta$. XYZ is the Simson line for $\theta°$.

Prove that $XZ = \frac{PC\ SinC}{Sin\theta}$.

To Prove : $XZ = \frac{PC\ SinC}{Sin\theta}$.

Construction : Join PA

Proof:

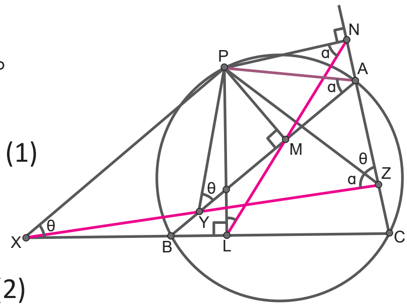


Figure : 34

$\angle PMA + \angle PNA = 90° + 90° = 180°$

$\Longrightarrow$ PMAN is concyclic.

$\Longrightarrow \angle PAM = \angle PNM = \alpha$ (say) --- (1)

$\angle PYA = \angle PZA = \theta$ (given)

$\Longrightarrow$PYZA is concyclic

$\Longrightarrow \angle PZY = \angle PAY = \alpha$ --------------- (2)

(1) & (2) $\rightarrow \angle PNM = \angle PZY = \alpha$

$\Longrightarrow \angle PNL = \angle PZX$ ------------------- (3)

$\angle PZN = \angle PXC = \theta$ (given)

ie exterior angle is equal to interior opposite angle of Quadrilateral PXCN.

$\Longrightarrow$ PXCN is concyclic

$\Longrightarrow \angle XPZ = 180° - \angle C$ ----------------(4)

$\angle PLC + \angle PNC = 90° + 90° = 180°$

$\Longrightarrow$ PLCZ is concyclic

$\Longrightarrow \angle LPN = 180° - \angle C$ -----------------(5)

(4)&(5) $\longrightarrow \angle LPN = \angle XPZ$ ------------- (6)

(3) & (6) $\longrightarrow \Delta LPN \sim \Delta XPZ$

$\Longrightarrow \frac{LP}{XP} = \frac{LN}{XZ} = \frac{PN}{PZ}$ -------------------(7)

But $\frac{PN}{PZ} = Sin\ \theta$ --------------------(8)

(7)& (8) $\longrightarrow \frac{LN}{XZ} = Sin\ \theta$

$\frac{PC\ SinC}{XZ} = Sin\theta.$

$XZ = \frac{PC\ SinC}{Sin\theta}$ ------------------------------ Proved.

*****

Theorem 8 : Simson line on a point P for ΔABC can never exceed its adjacent side.

Simson line with reference to a point P on the circumcircle of a ΔABC can never exceed its adjacent side.

In the figure:35, ΔABC is inscribed in the circle. P is a point on the minor arc AC. Perpendiculars are dropped from P to the three sides and the Simson line LMN is formed. Now, **LN can never exceed AC.**

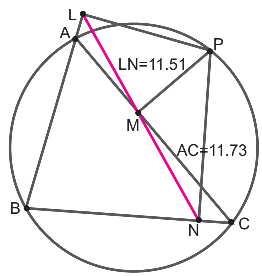


Figure : 35

Proof:

Length of Simson line LN = PB sin B

Length of side AC = 2R sin B

(PB length is always less than or equal to diameter)

Therefore, the length of this Simson line(for 90°) LN can never exceed the side AC.

*****

Theorem 9 : Equal Simson lines for triangles formed by any three vertices of a cyclic quadrilateral Theorem

Raja Climax has proved that for the triangles formed by any three vertices of a cyclic quadrilateral, the Simson lines drawn with reference to the point of fourth vertex will be equal.

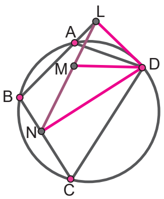


Figure - 1

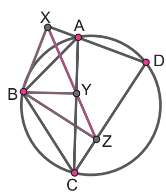


Figure - 2

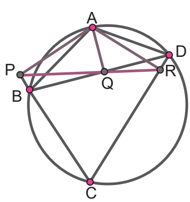


Figure - 3

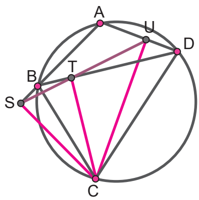


Figure - 4

In Figure - 1, ABCD is a cyclic quadrilateral. For ΔABC, Simson line LN is drawn treating D as a point on the circumference.

Similarly, in figure - 2, Simson line XZ is drawn to ΔADC treating B as a point on the circumference and Simson lines PR and SU are drawn to ΔBCD & ΔABD respectively treating A & C as point on circumference in figure - 3 and figure - 4 respectively.

Now, **the four Simson lines LN (figure-1), XZ (figure-2), PR (figure-3), & SU (figure-4), are equal in length.**

Proof: By theorem 5, In figure 1, LN = BD Sin B -----(1)

In figure 2, XZ = BD Sin D ------(2)

In a cyclic quadrilateral,

Opposite angles are supplementary .

∴ Sin B = Sin (180° – D) = Sin D, hence, by (1) & (2) LN = XZ.

Similarly, by theorem 5 In figure 3, PR = AC Sin C -------(3)

In figure 4, SU = AC Sin A -------(4)

∴ Sin A = Sin (180° – C) = Sin C, hence, by (3) & (4) PR = SU.

Similarly, we can prove LN = PR

Hence LN = PR = XZ =SU

*****

Theorem 9 : Simson line Congruency Theorem

ΔABC is inscribed in the circle. P is a point on the minor arc AB. PL, PM & PN are lines dropped from P to AC, AB & BC respectively such that ∠PLA = ∠PMA = ∠PNB = θ and LMN is the Simson line (for θ°). PL meets the circle at Z; PM produced and PN produced meet the circle at Y & X respectively. Now, ΔABC & Δ XZY are congruent.

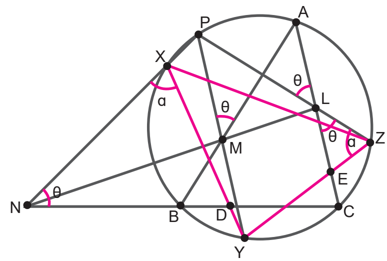


Figure : 36

This congruency occurs irrespective of the value of θ for all Simson lines. This can be called as the “Simson Line Congruency”.

Proof:

$\angle\mathrm{PLA} = \angle\mathrm{PMA} = \theta$

$\Rightarrow PMLA$ is a cyclic Quadrilateral

$\angle\mathrm{MPL} = \angle\mathrm{MAL} = \beta$ (say)

$\therefore \angle\mathrm{MPL} = \angle\mathrm{YPZ} = \angle\mathrm{YXZ} = \beta$

$\Rightarrow YZ = BC$ & $\angle\mathrm{BAC} = \angle\mathrm{YXZ}$

$\angle\mathrm{NXY} = \alpha$ (say), then $\angle\mathrm{PZY} = \alpha$

$\angle\mathrm{ELZ} = \theta, \angle\mathrm{LEZ} = \angle\mathrm{YEC} = 180° - (\alpha + \theta)$

$also \angle\mathrm{XDN} = \angle\mathrm{YDC} = 180° - (\alpha + \theta)$

$\Rightarrow YDEC$ is a cyclic Quadrilateral

$\Rightarrow \angle\mathrm{BCA} = \angle\mathrm{XYZ}$ & $AC = XZ$

By SAS congruency $\Delta ABC \cong \Delta XZY$ --------------- Hence proved.

*****

VI. NEW PROPERTIES OF 45°, 15°, 22.5°, 60° & 120° ANGLES

Raja Climax has also created certain new formulas in respect of the angles of 45°, 15° and 22.5°, when they are borne by triangles. Let us see them.

Theorem 10(A) : 45° Theorem:

If, in a triangle, the angle at one vertex is 45°, then, the distance between that vertex and its Orthocentre is equal to the side of the triangle opposite to that vertex.

Given :

In ΔABC, O is its orthocentre. $\angle BAC = 45°$

To Prove :

OA = BC

Construction :

Draw the altitudes BD & CE through O to AC and AB respectively.

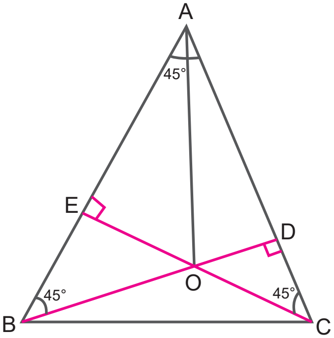


Figure : 37

Proof :

$\angle BAD = 45°, \therefore \angle DBA = 45°$

And BD = AD ---------------------- (1)

Similarly $\angle ECA = 45°$

Ie $\angle OCD = \angle COD = 45°$

∴ DC = DO --------------------- (2)

$\angle ODA = \angle CDB = 90°$------- (3)

From (1), (2) & (3) $\Delta ODA \cong \Delta CDB$

OA = BC -------------- Proved

*****

Theorem 10(B) : 15° Theorem

If, in a right ΔABC, right-angled at C,

$\angle B = 15^{\circ}$, then, $\boldsymbol{BC \times CA = \frac{AB^2}{4}}$

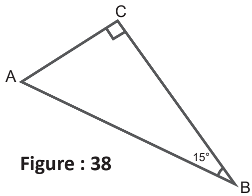


Figure : 38

Proof:

Construction:

Draw circumcircle of ΔABC
Drop a perpendicular from C (to AB) which meets AB at E and the circle at D.
Join AD and BD.
Mark O (on AB) as Circumcenter.
Join OC & OD.

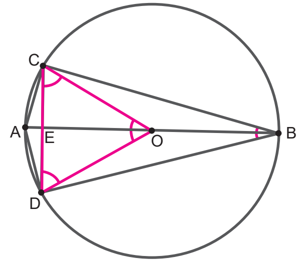


Figure : 39

$\because \angle CBD = 30^{\circ}, \quad \angle COD = 60^{\circ}$

∴OC = OD = CD = R, CE = $\frac{R}{2}$, AB = 2R

In ΔABC, Area of ΔABC $= \frac{1}{2} \times AC \times CB = \frac{1}{2} \times AB \times CE$

$\Rightarrow$ AC x CB = EC x AB = $(\frac{R}{2})$ x 2R

AC x CB = $R^2 = \frac{AB^2}{4}$

*****

Theorem 10(C) : 22.5° Theorem

In a right angle triangle, if one angle is 22.5°, then, the product of the sides forming the right angle equals $\sqrt{2}$ times the square of the circumradius.

ie $\boldsymbol{BC \times CA} = \sqrt{2}R^2$.

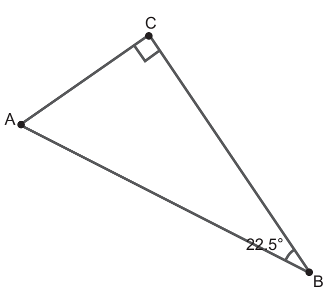


Figure : 40

Proof:

Draw circumcircle of ΔABC.

Drop a perpendicular from C (to AB) which meets AB at E and the circle at D.
Join AD and BD.
Mark O (on AB) as Circumcenter. Join OC & OD.

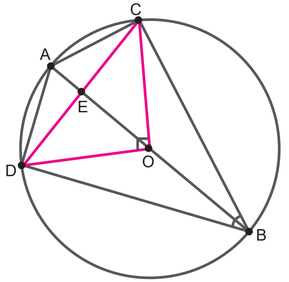


Figure : 41

$\because \ \angle CBD = 45^\circ, \ \angle COD = 90^\circ$

ΔODC is isosceles right triangle.

OC = R = OD,

$\Rightarrow$CD = $\sqrt{2}$R & CE = $\frac{R}{\sqrt{2}}$

In ΔABC, Area of ΔABC = $\frac{1}{2} \times AC \times CB \ = \frac{1}{2} \times AB \times CE$

$= 2R \times \frac{R}{\sqrt{2}}$

(AC x CB) = $\sqrt{2}$ R^2

*****

Theorem 10(D) : 60° Theorem

In ΔABC, ∠A = 60°.

Prove: $AB^2 + AC^2 - (AB \times AC) = 3R^2$,

where R is the circumradius of ΔABC.

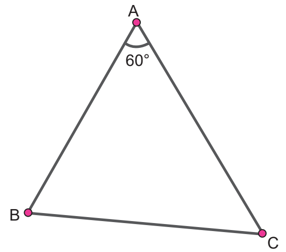


Figure : 42

Proof :

By Cosine rule,

$a^2 = b^2 + c^2 - 2bc\ CosA$.

$BC^2 = AC^2 + AB^2 - 2AC \ \times AB\ Cos60°$

$BC^2 = AC^2 + AB^2 - 2AC \times \ AB \ \times \frac{1}{2}$ ---------(I)

BC = 2R Sin60° = $2R \times \frac{\sqrt{3}}{2} = \sqrt{3}R$

Substituting BC in (I) $AB^2 + AC^2 - (AB \ \times AC) \ = 3R^2$

*****

Theorem 10(E) : 120° Theorem

In triangle ABC, ∠A = 120°. AD is angle bisector.

Then, AD = $\frac{(AB\times AC)}{(AB+AC)}$.

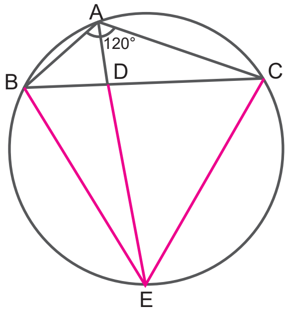


Figure : 43

Given :

In $\Delta ABC, \angle A = 120°$. AD is the angle bisector.

To Prove : $AD = \frac{(AB\times AC)}{(AB+AC)}$

Construction :

Draw the circumcircle for ΔABC. Produce AD to meet the circumcircle at E. Join BE & CE.

Proof:

$\angle BEC = 180° - 120° = 60°$ -------------------------- (1)

Since AE is angle bisector,

BE = EC ------------------------(2) (Equal angles have equal chords)

(1) & (2) → ΔBEC is equilateral.

⟹ AB + AC = AE -----------(3) (Applying Ptolemy's Theorem)

(2) → E is the midpoint of major arc BC.

⟹ Applying Sides and Angle Bisector Theorem, (available in this books in page No: 66)

$AB \times AC = AE \times AD$

⟹ AD = $\frac{AB\times AC}{AE}$ --------------------------- (4)

(3) & (4) →AD = $\frac{AB\times AC}{AB+AC}$ ------------------------ **Proved**

VII. ADVANCEMENT ON THE PYTHAGORAS THEOREM

The Pythagoras Theorem speaks only about the right triangles. It is silent about the other triangles. But there is a relationship among the squares of the three sides of any triangle. This relationship is explained in the Advanced Pythagoras Theorems by **Raja Climax**.

Theorem 11 (A) : Theorem for Acute Angled Triangles:

(In Geometric terms)

ABC is an acute angled triangle. AD is an altitude and O is the Orthocentre,
Then, $AB^2 + AC^2 = BC^2 + 2AD \times AO$.

Given:

In acute angled $\Delta ABC, AD$ is an altitude and O is the Orthocentre.

To Prove: $AB^2 + AC^2 = BC^2 + 2 \times AD \times AO$

Construction:

Draw the circumcircle of ΔABC.

Produce AD to meet the circumcircle at E.

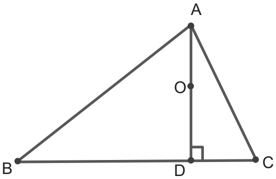


Figure : 44

Proof:

$AB^2 = AD^2 + BD^2$ ------------------------- (1)

$AC^2 = AD^2 + CD^2$ -------------------------- (2)

(1) + (2) → $AB^2 + AC^2 = BD^2 + CD^2 + 2AD^2$

$= (BD + CD)^2 - 2 \times BD \times CD + 2AD^2$

$= BC^2 - 2 \times AD \times DE + 2AD^2$

$(\because BD \times DC = AD \times DE)$

$= BC^2 + 2AD(AD - DE)$

$= BC^2 + 2 \times AD \times AO$

($\because OD = DE$. (Proved))

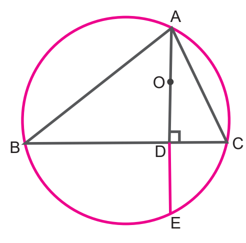


Figure : 45

Importance of the Advance Pythagoras Theorem (Acute) :

This theorem helps us to find the relationship among the sides of an acute angled triangle and the pieces of the altitudes drawn to them.

THEOREM 11 (b) : Theorem For Obtuse Angled Triangles

(In Geometric terms)

ΔABC is obtuse angled at vertex A. AD is an altitude and O is orthocentre.

Then $\boldsymbol{AB^2 + AC^2 = BC^2 - 2 \times AD \times AO}$

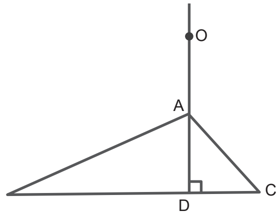


Figure : 46

Given:

In $\Delta ABC, \angle A$ is obtuse. AD is an altitude and O is its Orthocentre.

To Prove:

$AB^2 + AC^2 = BC^2 - 2AD \times AO$

Construction:

Draw the circumcircle of ΔABC.

Produce AD to meet the circumcircle at E.

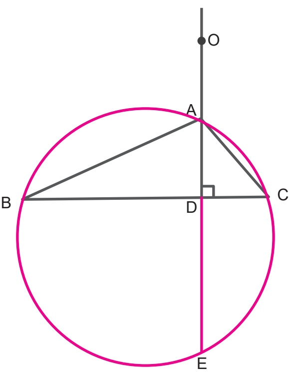


Figure : 47

Proof:

$AB^2 = AD^2 + BD^2$ ----------------- (1)

$AC^2 = AD^2 + CD^2$ ----------------- (2)

(1) + (2) →

$AB^2 + AC^2 = BD^2 + CD^2 + 2AD^2$

$= (BD + CD)^2 + 2AD^2 - 2 \times BD \times CD$

$= BC^2 + 2AD^2 - 2 \times AD \times DE$ $(\because BD \times DC = AD \times DE)$

$= BC^2 - 2AD(DE - AD)$

$= BC^2 - 2 \times AD \times AO$ $(\because OD = DE)$ **------Proved**

Importance of the Advance Pythagoras Theorem (Obtuse) :

This theorem helps us to find the relationship between the sides of obtuse angled triangle and the pieces of the altitude drawn to it.

N.B:

When the ΔABC is right angled at A,

$AB^2 + AC^2 = BC^2$ (Pythagoras property)

$(\because AO = 0 \text{ and } \therefore AO \times AD = 0)$

*****

VIII. THE CONCURRENCY OF CEVIANS

Theorem-12 : The Concurrency Theorem

The Cevas Theorem speaks about the concurrency of cevians. It tells what happens along the perimeter of the triangle, when three cevians are concurrent. But here, the Concurrency Theorem (which has been developed by **Raja climax**) tells us what happens to the cevians inside the triangle.

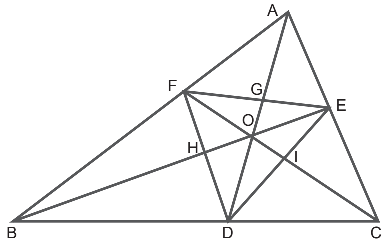


Figure : 48

In ΔABC, D, E and F are base points on sides BC, AC and AB respectively such that AD, BE and CF (Cevians) are concurrent at O. FE and AD intersect at G; FD and BE at H; and DE and CF at I, then,

$$\frac{AG}{GO} = \frac{AD}{DO}\ ;\ \frac{BH}{HO} = \frac{BE}{EO}\ ;\ \text{and}\ \frac{CI}{IO} = \frac{CF}{FO}$$

This is the Concurrency Theorem.

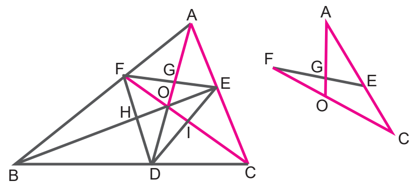


Figure : 49

Proof :

For ΔAOC, EGF is a transversal cutting the sides AC, AO and CO at E, G and F respectively.

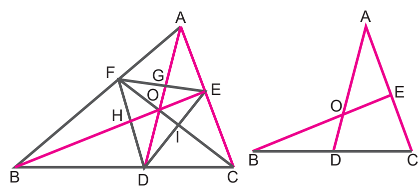


Figure : 50

∴ As per Menelaus Theorem,

$$\frac{CE}{EA}\ X\ \frac{AG}{GO}\ X\ \frac{OF}{FC} = 1 \text{------ (1)}$$

For ΔAOE, CDB is a transversal cutting its sides AE, AO & OE at C, D and B respectively.

∴ As per Menelaus Theorem,

$$\frac{AC}{CE}\ X\ \frac{EB}{BO}\ X\ \frac{OD}{DA} = 1 \text{ ------------------ (2)}$$

Multiply (1) & (2)

$$\frac{\cancel{CE}}{EA} \times \frac{AG}{GO} \times \frac{OF}{FC} \times \frac{AC}{\cancel{CE}} \times \frac{EB}{BO} \times \frac{OD}{DA} = 1$$

ie $\left(\frac{AG}{GO} \times \frac{OD}{DA}\right)$ $=\left(\frac{EA}{AC} \times \frac{CF}{FO} \times \frac{OB}{BE}\right)$ -------- (3)

ForΔCOE, BFA is a transversal cutting its sides EO, CO and CE at B, F & A respectively.

∴ As per Menelaus Theorem,

$\frac{EA}{AC} \times \frac{CF}{FO} \times \frac{OB}{BE} = 1$ ----- (4)

From (3) & (4)

$\frac{AG}{GO} \times \frac{OD}{DA} = 1$

ie $\frac{AG}{GO} = \frac{AD}{DO}$

Similarly, we can prove

$\frac{BH}{HO} = \frac{BE}{EO}$ and $\frac{CI}{IO} = \frac{CF}{FO}$

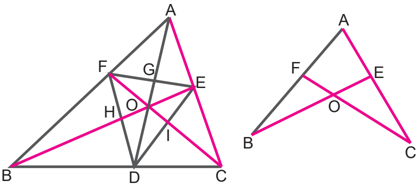


Figure : 51

‘CONCURRENCY THEOREM’ ----------------- Proved.

No. of Riders framed by Raja Climax based on this theorem is more than 25. (These riders are available in www.maxgeometricmaths.co.in)

Importance of the Concurrency Theorem:

1. Helps us to find the ratios of the segments of cevians.

2. Helps us to find the ratios of corresponding areas of the triangles having segments as sides in the cevians.

3. It teaches the easiest method to divide a straight line internally and externally in the same ratio.

4. It proves certain impossibilities. For example, it proves that "No two cevians drawn inside a triangle can bisect each other”.

*****

Theorem 13: The Unity Pieces Theorem

In ΔABC, AD, BE and CF are cevians concurrent at ‘O’.

AD and FE intersect at G, BE and FD at ‘H’ and CF and DE at I.

Prove that 1. $\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = 1$

2. $\frac{OG}{GA} + \frac{OH}{HB} + \frac{OI}{IC} = 1$

Proof:

$\frac{AD}{DO} = \frac{\Delta ABD}{\Delta OBD} = \frac{\Delta ACD}{\Delta OCD}$ (Δmeans area of the triangle)

= $\frac{\Delta ABD+\Delta ACD}{\Delta OBD+\Delta OCD}$ $\left(\frac{sum\ of\ the\ numerators}{sum\ of\ the\ denominators}\right)$

= $\frac{\Delta ABC}{\Delta OBC}$

$\frac{OD}{AD} = \frac{\Delta OBC}{\Delta ABC}$ -------------------------------- (1)

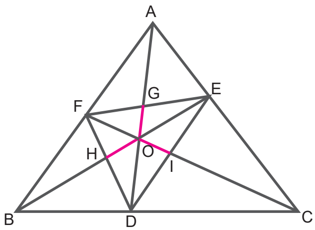


Figure : 52

Similarly, we can prove that

$\frac{OE}{BE} = \frac{\Delta OCA}{\Delta ABC}$ -------------------------------- (2)

And $\frac{OF}{CF} = \frac{\Delta OBA}{\Delta ABC}$ ------------------------ (3)

(1) +(2) + (3) ⟶

$$\frac{OD}{\mathrm{AD}} + \frac{OE}{EB} + \frac{OF}{CF} = \frac{\Delta OBC}{\Delta ABC} + \frac{\Delta OCA}{\Delta ABC} + \frac{\Delta OBA}{\Delta ABC} = \frac{\Delta ABC}{\Delta ABC} = 1$$

$\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = 1$ ------------------------------- (4)

But, As per **'CONCURRENCY THEOREM'**,

$\frac{AG}{\mathrm{GO}} = \frac{AD}{DO}$, $\frac{BH}{\mathrm{HO}} = \frac{BE}{EO}$ and $\frac{CI}{\mathrm{IO}} = \frac{CF}{FO}$

ie. $\frac{OG}{\mathrm{GA}} = \frac{OD}{AD}$, $\frac{OH}{\mathrm{HB}} = \frac{OE}{BE}$ and $\frac{OI}{\mathrm{IC}} = \frac{OF}{CF}$

$\therefore \frac{OG}{GA} + \frac{OH}{HB} + \frac{OI}{IC} = \frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF}$ ------------------ (5)

(4) & (5) ⟶

$\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = 1$ ------------------------ Proved

$\frac{OG}{GA} + \frac{OH}{HB} + \frac{OI}{IC} = 1$ ------------------- Proved

*****

Corollary:

$$\frac{\Delta OFE}{\Delta AFE} + \frac{\Delta OFD}{\Delta BDF} + \frac{\Delta ODE}{\Delta DCE} = 1$$

No of Riders framed by Raja Climax based on this theorem is 11.

(These riders are available in www.maxgeometricmaths.co.in)

*****

IX. THE INFINITELY PROLONGED CONCURRENCY

Theorem 14 : The Endless Concurrency Theorem: (EC Theorem)

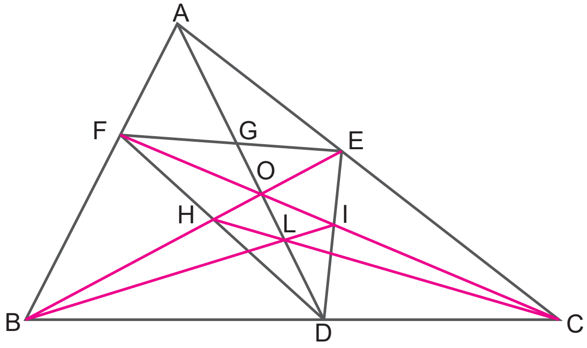


Figure : 53

In the figure:53, AD, BE & CF are cevians concurrent at O. AD and FE intersect at G, BE and FD at 'H' and CF and DE at I.

If BI & CH are joined, their point of intersection falls on OD only. ie, BI, CH & OD are also concurrent, say at L. This is an additional concurrency at L. This has been proved Geometrically by Raja Climax. There is also an interesting additional property on this new concurrency.

ie, $\frac{OG}{GA} + \frac{OL}{LD} = 1$. Raja climax has proved this also.

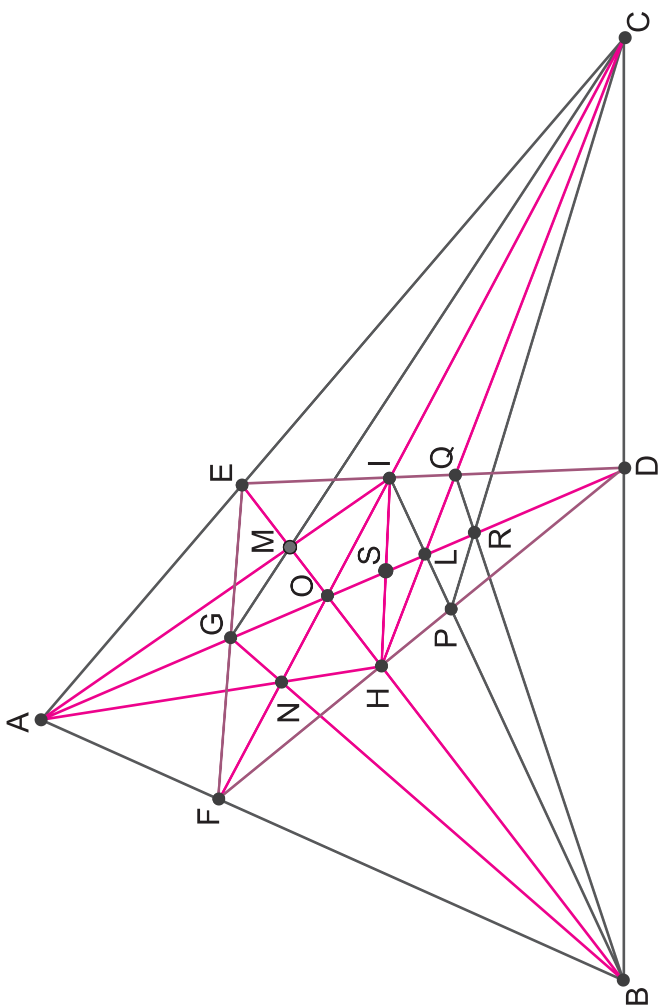


Figure : 54

This concurrency can go on repeating. ie, Suppose BI & DH meet at P and CH & DI meet at Q, then BQ & CP are also concurrent with LD say at R. (Here also, if we join IH to cut OD at S, then $\frac{LS}{SO}+\frac{LR}{RD}=1$). This concurrency process goes on endlessly. So this theorem is called the **"Endless Concurrency Theorem".**

Given:

In ΔABC, AD, BE & CF are cevians concurrent at O. Join DE, EF & FD. AD & FE intersect at G, BE & FD intersect at H and CF & ED intersect at I.

Now, prove:

i. BI, CH & OD are concurrent (Say at L)
ii. CG, AI & OE are concurrent (Say at M)
iii. BG, AH & OF are concurrent (Say at N)
iv. $\frac{OG}{GA}+\frac{OL}{LD}=1, \frac{OH}{HB}+\frac{OM}{ME}=1$ & $\frac{OI}{IC}+\frac{ON}{NF}=1$
v. $\frac{OL}{LD}+\frac{OM}{ME}+\frac{ON}{NF}=2$

Proof:

For $\Delta ABO, FHD$ is a transversal.

$\therefore \frac{AF}{FB}\times\frac{BH}{HO}\times\frac{OD}{AD}=1$ [Menelaus Theorem]

$\Rightarrow \frac{BH}{HO}=\frac{AD}{OD}\times\frac{FB}{AF}$ --------------------------------(1)

For $\Delta ACO, EID$ is a transversal

$\therefore \frac{CE}{EA}\times\frac{AD}{DO}\times\frac{OI}{IC}=1$ (Menelaus theorem)

$\Rightarrow \frac{OI}{IC}=\frac{AE}{EC}\times\frac{OD}{AD}$ ---------------------------------------(2)

(1) × (2) → $\frac{BH}{HO}\times\frac{OI}{IC}=\frac{AD}{DO}\times\frac{BF}{FA}\times\frac{AE}{EC}\times\frac{OD}{AD}$

$\frac{BH}{HO}\times\frac{OI}{IC}=\frac{BF}{FA}\times\frac{AE}{EC}$ -------------------------------------(3)

In ΔABC, the cevians AD, BE & CF are concurrent at O.

$\therefore \frac{BF}{FA}\times\frac{AE}{EC}\times\frac{CD}{BD}=1$ (Ceva's Theorem) ----------(4)

(3) & (4) → $\frac{BH}{HO}\times\frac{OI}{IC}\times\frac{CD}{BD}=1$

∴ As per the converse of Ceva's Theorem, in ΔOBC, the Cevians BI, CH & OD will be concurrent, say at L. Similarly, we can prove that in ΔLBC, the Cevians BQ, CP & LD are also concurrent say at R and so on.
As per the Unity Pieces Theorem, (invented by Raja Climax in 2019 and published by him in page 6 of his book "The Geometry of Concurrency" Vol-II),

In the $\Delta ABC, \frac{OG}{GA} + \frac{OH}{HB} + \frac{OI}{IC} = 1$

$\Rightarrow \frac{OH}{HB} + \frac{OI}{IC} = 1 - \frac{OG}{GA}$ -------------------------------- (1)

For ΔOBD, HLC is a Transversal.

$\therefore \frac{DL}{LO} \times \frac{OH}{HB} \times \frac{BC}{CD} = 1$ (Menelaus Theorem)

$\Rightarrow \frac{OH}{HB} = \frac{OL}{LD} \times \frac{CD}{BC}$ ------------------------------------- (2)

For ΔOCD, ILB is a Transversal.

$\therefore \frac{DL}{LO} \times \frac{OI}{IC} \times \frac{BC}{BD} = 1$ (Menelaus Theorem)

$\Rightarrow \frac{OI}{IC} = \frac{OL}{LD} \times \frac{BD}{BC}$ -------------------------------------- (3)

$(2) + (3) \rightarrow \frac{OH}{HB} + \frac{OI}{IC} = \frac{OL}{LD}\left[\frac{BD + DC}{BC}\right] = \frac{OL}{LD} \times \frac{BC}{BC}$

$\frac{OH}{HB} + \frac{OI}{IC} = \frac{OL}{LD}$ --- (4)

(1) & (4) $\rightarrow \frac{OL}{LD} = 1 - \frac{OG}{GA}$

$\frac{OL}{LD} + \frac{OG}{GA} = 1$ ------------------------------------- Proved.

Similarly we can prove $\frac{OH}{HB} + \frac{OM}{ME} = 1$ & $\frac{OI}{IC} + \frac{ON}{NF} = 1$

$\frac{OL}{LD} + \frac{OM}{ME} + \frac{ON}{NF} = \left(1 - \frac{OG}{GA}\right) + \left(1 - \frac{OH}{HB}\right) + \left(1 - \frac{OI}{IC}\right)$

$= 3 - \left(\frac{OG}{GA} + \frac{OH}{HB} + \frac{OI}{IC}\right)$

$= 3 - 1 = 2$ (By unity pieces theorem)

$\boldsymbol{\frac{OL}{LD} + \frac{OM}{ME} + \frac{ON}{NF} = 2}$ **-------------------Hence proved.**

X - THE ORTHOCENTRE

For any point inside a circle, we can inscribe a triangle, so that the said point becomes its Orthocenter. Let the point inside the circle be O. Now, let us see how a triangle can be drawn inside the circle so that O is its orthocentre under the Orthocentre Theorem, developed by Raja Climax.

Theorem 15 : The Orthocentre Theorem *(In Geometric terms)*

O is any point inside the circle. *A* ΔABC can be inscribed in the circle so that O becomes its Orthocenter.

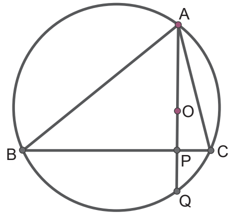


Figure : 55

Construction:

Let the point inside the circle be O. Now, let us see how a triangle can be drawn inside the circle so that O is its orthocentre.

Construction:

1. Draw a chord through the point O. Mark the end points of the chord as A, Q as shown in the picture.
2. Mark the midpoint of OQ as P and construct the perpendicular bisector for OQ through P.
3. Mark the end points of it as B & C. (Now, chord BC is perpendicular to AQ)
4. Join AB & AC.
   Now, O is the Orthocenter of ΔABC.

INFERENCE:

Infinite number of triangles can be inscribed in the circle for which the same point O becomes the Orthocenter of all those triangles. For all these triangles, the Twelve-Points Circle is the same.

To construct a different acute triangle with the same Orthocenter:

1. O is the orthocenter for acute ΔABC. Draw circumcircle for ΔABC.
2. Draw a chord QQ’ through O. Construct perpendicular bisector for OQ’ which meets the circle at P & R.
3. Join PQ and QR, ΔPQR is the required triangle.
4. Now ΔABC & ΔPQR have same orthocenter at O.
5. Similarly, we can construct many triangles with the same orthocenter.

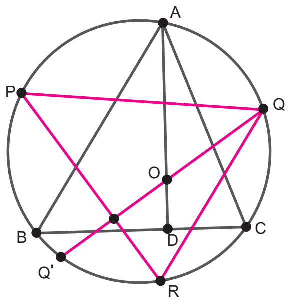


Figure : 56

To construct a different right triangle with the same orthocenter:

1. O is the orthocenter for right ΔOAB. Draw circumcircle for ΔOAB. Circumcenter C is at midpoint of AB.
2. Draw a chord PQ through C. Join PO and QO, ΔPQO is the required triangle.
3. Now ΔABO & ΔPQO have same orthocenter O.
4. Similarly, we can construct many triangles with the same orthocenter.

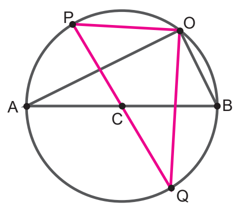


Figure : 57

To construct a different obtuse triangle with the same orthocenter:

1. O is the orthocenter for obtuse ΔABC. Draw circumcircle for ΔABC.
2. Draw a chord QQ' through O. Construct perpendicular bisector for OQ’ which touches the circle at P & R.
3. Join PQ and QR, PQR is the required triangle.
4. Now ΔABC & ΔPQR have same orthocenter.
5. Similarly we can construct many triangles with the same orthocenter.

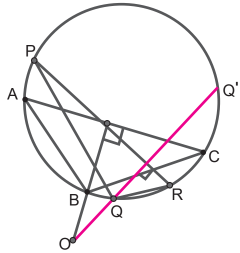


Figure : 58

Importance of the Orthocentre Theorem:

1. Before this theorem was developed by Raja Climax, if we fix a point inside a circle and try to draw a triangle so that the point fixed becomes the orthocentre of the triangle, we could not. But now, with the help of the Orthocentre Theorem, we can draw a triangle after fixing its orthocentre.

*****

Theorem 16 : The Double Similarity Theorem:
(In textual terms)

The Orthic triangle of a triangle and the triangle formed by the meeting points of its altitudes with its circumcircle are similar to each other with their sides in the ratio of 1: 2.

(In Geometric terms)

AD, BE & CF are the altitudes of Δ ABC with Orthocentre at O and with Orthic triangle, Δ DEF. AD, BE & CF are produced to meet the circumcircle of Δ ABC at P,Q & R respectively. Prove that Δ DEF and Δ PQR are similar & $\frac{DE}{PQ} = \frac{EF}{QR} = \frac{FD}{RP} = \frac{1}{2}$.

Proof:

Given:-

AD, BE & CF are the altitudes of Δ ABC with Orthocentre at O and with Orthic triangle DEF. AD, BE & CF are produced to meet the circumcircle of Δ ABC at P,Q & R respectively.

To Prove:-

Δ DEF ~ Δ PQR &

$\frac{DE}{PQ} = \frac{EF}{QR} = \frac{FD}{RP} = \frac{1}{2}$.

Proof:-

First let us prove that PQ ∥ DE andPQ = 2DE that OD=DP & OE=EQ. (Proved)

∴ DE becomes the line joining the mid points of OP & OQ.

∴ DE ∥ PQ & PQ = 2DE ------ *Proved.*

Similarly, we can prove that QR ∥ EF and QR = 2EF and that RP ∥ FD & RP=2FD.

∴ $\Delta DEF \sim \Delta PQR$

$\frac{\mathbf{DE}}{\mathbf{PQ}} = \frac{\mathbf{EF}}{\mathbf{QR}} = \frac{\mathbf{FD}}{\mathbf{RP}} = \frac{\mathbf{1}}{\mathbf{2}}$ **-------- Proved.**

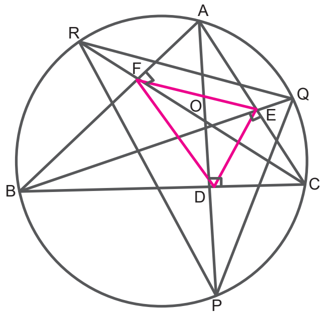


Figure : 59

Importance of the Double Similarity Theorem:

The earlier Mathematicians had stated that the orthic triangle is similar to the triangle formed by the meeting points of its altitudes with its circumcircle. It is Raja Climax who proved that the ratio of the sides of the orthic triangle to that of the latter triangle is 1:2 and the area of the orthic triangle is one fourth of area of the latter triangle.

*****

Theorem 17 : The Ex-Centre Theorem:

(In textual terms)

When a triangle is Obtuse-angled, its Orthocentre is not only the excentre of its Orthic triangle, but also the excentre of the triangle formed (opposite to the Obtuse vertex) by the meeting points of its altitudes with its circumcircle.

(In Geometric terms)

ABC is a Δ Obtuse-angled at A. Its altitudes AD (drawn to BC), BE (drawn to CA produced) and CF (drawn to BA produced) intersect its circumcircle at P,Q & R respectively. (with concurrency at ‘O’, the Orthocentre). Prove that ‘O’ is the excentre of ΔPQR [formed opposite to vertex P] and also orthic triangle ΔDEF.

Given : -

ABC is a Δ Obtuse-angled at A. Its altitudes AD (drawn to BC), BE (drawn to CA produced) and CF (drawn to BA produced) intersect its circumcircle at P,Q & R respectively. (with concurrency at 'O', the Orthocentre).

To prove:-

'O' is the excentre of ΔPQR & ΔDEF [formed opposite to vertex P].

Construction:-

Produce CE & PQ to meet at L. Produce PR to K.

Proof:

$\angle LQO = \angle BQP = \angle BAP = \angle OAF$

i.e, $\angle LQO = \angle OAF$ ....(1)

DAFC is concyclic. [$\because \angle D + \angle F = 180^{\circ}$]

$\therefore \angle OAF = \angle RCD$ ....(2)

BQRC is concyclic.

$\therefore \angle OQR = \angle BCR = \angle RCD$

i.e, $\angle OQR = \angle RCD$ ....(3)

From (1), (2) & (3) → $\angle LQO = \angle OQR$

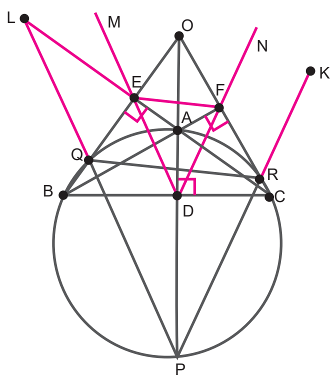


Figure : 60

i.e, QO is the angle bisector of $\angle LQR$

Similarly, we can prove that RO is the angle bisector of $\angle QRK$

∴ O is the excentre of triangle PQR ----------Proved.

To prove that O is the excentre for ΔDEF

$\angle OFE = \angle FEC + \angle ECF$

$\angle DAC = \angle AOC + \angle ACF$

i.e, $\angle OFE = \angle DAC$ ----------------------------------(4)

$\angle OFN = \angle DFC = \angle DAC$

From (4) we have $\angle DAC = \angle OFE = \angle OFN$

Hence $\angle OFN = \angle OFE$

i.e, FO is the angle bisector of $\angle NFE$

Similarly, we can prove that OE is the angle bisector of $\angle MEF$

∴ O is the excentre of triangle DEF ----------Proved.

*****

Importance of the Ex-center Theorem:

The earlier Mathematicians had stated that the orthocentre of an acute triangle becomes the incenter of its orthic triangle. But they were silent about the orthocentre of obtuse-angled triangle with reference to its orthic triangle. It is Raja Climax who proved that the orthocentre of obtuse angled triangle is the ex-center for the orthic triangle and also ex-center for the triangle formed by the altitudes produced to the circumcircle. This truth is given here as the Ex-center Theorem.

XI. THE ALTITUDES

Theorem 18 : The Altitude Segments Theorem

An altitude has two segments, one segment joins the orthocentre to the vertex. The other segment joins orthocentre to its foot. Here, Raja Climax tells that the second segment can be calculated from the altitude and its side. This is stated as the Altitude Segments Theorem.

The Altitude Segments Theorem:

In a ΔABC, If AD, BE and CF are the altitudes through the vertices A,B and C respectively and O is its orthocentre, then

- OD = $\frac{(BD \times DC)}{AD}$
- OE = $\frac{(CE \times EA)}{BE}$
- OF = $\frac{(AF \times FB)}{CF}$

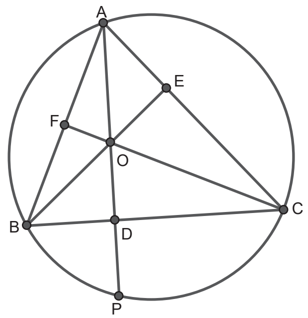


Figure : 61

Proof:

We know that $BD \times DC = AD \times DP$

But we have already proved OD = DP

⟹ BD x DC = AD x OD

⟹ **OD = $\frac{(BD \times DC)}{AD}$ -------- Proved**

Similarly, we can prove remaining 2 results of this theorem.

Importance of the Altitude Segments Theorem:

Helps in calculating the orthic distance (distance between orthocenter and vertex) and distance between the orthocenter and the base.

*****

Theorem 19 : The Sides And Altitudes Theorems (SAT Theorems)

The sides and the altitudes of a triangle help us study many further concepts. The area of the triangle, the Orthocentre of the triangle, the concyclic points of the triangle, etc., are some of those concepts.

SAT Theorem 1:

Now, Raja Climax has discovered that the feet of the altitudes on any two sides help us find out the measurement of the third side. This is done by the "Sides and Altitudes Theorem 1" (SAT Theorem 1) formulated by him. Let us explain it diagrammatically. In the following figure:61, ABC is a triangle and BE and CF are its altitudes (E and F are the feet of those altitudes.)

Now, the Sides and Altitudes Theorem says,

(ABxBF + ACxCE) = BC^2 ---------------------- (I)

& (BExBO + CFxCO) = BC^2 ---------------------- (II)

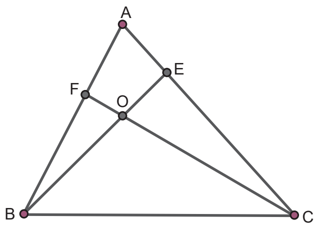


Figure : 62

Proof for (I) above:

Given:

In ΔABC, AD, BE & CF are altitudes and O is its orthocentre.

Prove: $(\boldsymbol{AB} \times \boldsymbol{BF}) + (\boldsymbol{AC} \times \boldsymbol{CE}) = \boldsymbol{BC^2}$

Proof:

ΔABD & ΔCBF are similar.

$$\Longrightarrow \frac{AB}{BC} = \frac{BD}{BF}$$

$\Longrightarrow$ AB $\times BF = BC \times BD$ ---------------(1)

ΔCAD & ΔCBE are similar.

$\Longrightarrow$AC×CE = BC × CD ---------------------(2)

$(1) + (2) \rightarrow$

$(AB \times BF) + (AC \times CE)$ = BC (BD +CD) = BC^2 --------------Proved.

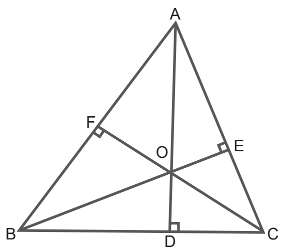


Figure : 63

Proof for (II) above:

Given:

In ΔABC, AD, BE & CF are altitudes and O is its orthocentre.

Prove: (BExBO + CFxCO) = BC^2

Proof:

AFOE is concyclic.

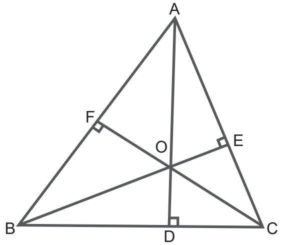


Figure : 64

⇒ AB x BF = BE x BO --------------------(1)

& AC x CE = CF x CO --------------------(2)

(1) + (2) →

(AB x BF) + (AC x CE) = (BE x BO) + (CF x CO) -------(3)

By SAT Theorem 1 - (I) , it was proved that

$(AB \times BF) + (AC \times CE) = BC^2$ --------------(4)

(3) & (4) →

(BE x BO) + (CF x CO) =BC^2 ------------------ Proved

If we know the measurements of AB, BE, AC & CD, within no time we can find out the measurement of BC using the identity (I) above. Also, if we know the measurements of BD, BO, CE & CO, we can easily find out the measurement of BC using the identity (II) above. In the identity (I), only the two sides (AB & AC) and their segments (BE & CD) are used to find the third side BC. In the identity (II), only the two altitudes (BD & CE) and their segments BO & CO are used to find the measurement of BC. Hence, this theorem is named as "The Sides and Altitudes Theorem 1" or shortly "The SAT Theorem 1".

SAT Theorem 2:

The above theorem can also be re-stated with reference to the diameter of a semi-circle and the chords shot from the ends of the diameter. ie This theorem can be employed to explain the relationship between the diameter of a semicircle and its chords. Look at the following diagram. Here, BC is the diameter of the semicircle and BA and CA are chords meeting externally at A. BE and CF are altitudes for ΔABC (∠BFC & ∠BEC are right angles because they are angles in the semicircle).

Now therefore, as per the SAT Theorem 1,

$(AB \times BF) + (AC \times CE) = BC^2 = d^2$ where d is the diameter, and

also **$(BE \times BO) + (CF \times CO) = BC^2 = d^2$**

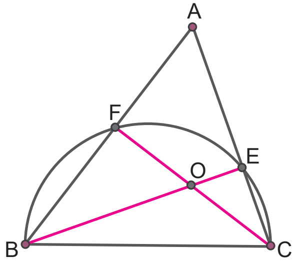


Figure : 65

Given:

In ΔABC, AD, BE & CF are altitudes and O is its orthocentre.

To prove: (BE x BO) + (CF x CO) = BC^2

Proof:

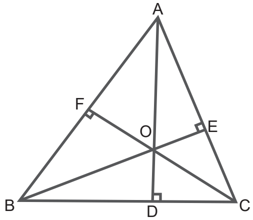


Figure : 66

AFOE is concyclic.

⇒ AB x BF = BE x BO -------------------(1)

& AC x CE = CF x CO -------------------(2)

(1) + (2) →

(AB x BF) + (AC x CE) = (BE x BO) + (CF x CO) --------------(3)

In the previous theorem (SAT Theorem 1), it was proved that

$(AB \times BF) + (AC \times CE) = BC^2$ --------------(4)

(3) & (4) →

(BE x BO) + (CF x CO) =BC^2 ------------------ Proved

Now, it is clear that if the measurements of the segments of any two chords starting from the ends of a diameter are known, we can find out the measurement of the diameter.

Using the SAT Theorems, many new Geometrical riders can be framed and solved. A few are given below:

Rider:1

In the figure:67, ADCB is a semicircle. $\angle BAC = 15°$. E is a point on BC produced such that BE = AC. AE meets the circle at D.

Prove: $(EA \times AD) = 3R^2$, where R is the circumradius.

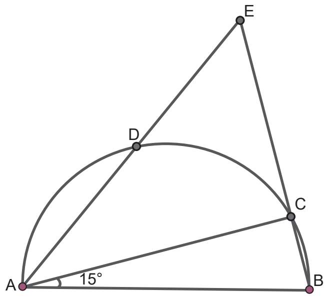


Figure : 67

Rider:2

In ΔABC, $\angle A = 45°$. Its altitudes BD & CE meet at O.

Prove: $(AB \times BE) + (AC \times CD) = AO^2$

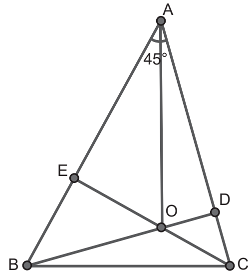


Figure : 68

Rider: 3

In the figure:69, M is the midpoint of the line segment AB and MC is its perpendicular bisector. BD is drawn perpendicular to AC. Given that AC=2BD. Find the measurement of $\angle A$.

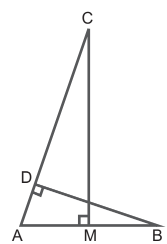


Figure : 69

Rider: 4

AB is a line segment with midpoint C. A semi circle is mounted on AB. D is a point above the semicircle such that AD = AB. AD & BD meet the semicircle at E & F respectively. EF is joined.

Prove: $EF^2 = (EC \times ED)$.

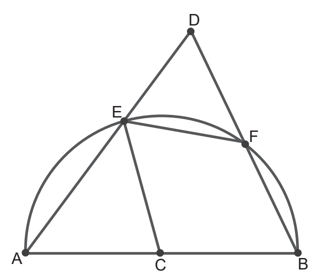


Figure : 70

Rider: 5

AB is a line segment with a semicircle mounted on it. C is a point on the semicircle such that chord BC is $\frac{1}{4}$AB. AC is joined and F is a point on BC produced such that BF = AB. D is a point on the semicircle such that ∠ABD = 15°. K is a point on AD produced such that AK = BD. AF & BK meet the circle at J & L respectively. Prove: $2JF^2$ = BK x BL.

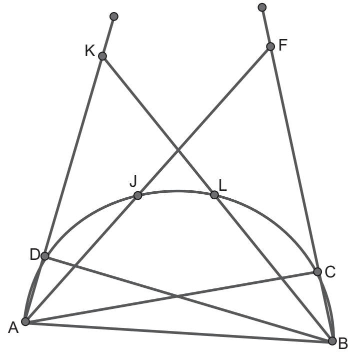


Figure : 71

Rider 6:

O is the Orthocentre of ΔABC. CD is the altitude from C to AB. L, M & N are the midpoints of BC, AO & BO respectively. DL & MN meet at E.

Prove: (LD x LE) + (MN x ME) = R^2, where R is the circumradius of ΔABC.

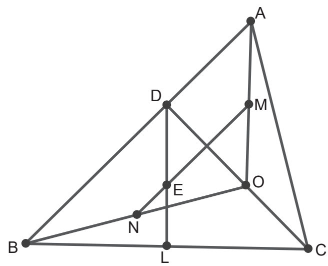


Figure : 72

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XII. THE BISECTING CHORDS INSIDE A CIRCLE

When two chords intersect inside or outside a circle, the product of one's segments is equal to the product of the other's segments. ie, If AB & CD are two chords intersecting at O either inside or outside a circle, then, AO x OB = CO x OD. This is the main result so far known to the Geometric world. But now, Raja Climax has discovered certain special situations where the intersecting chords give some interesting results. (One Chord bisecting the other chord.)

Theorem 20: The Bisecting Chord Theorem (The Saikumar's Theorem)

This theorem was developed jointly by Raja Climax and Sai Prasath Kumar, an eminent Mathematician from Andra Pradesh, India. Hence, this Theorem is called as the Saikumar's Theorem also.

When a chord is bisected by another chord (The bisecting chord), a relationship arises between the bisecting chord and the quadrilateral formed by the ends of the two chords. This relationship is explained by this theorem

The Saikumar Theorem (The Bisecting Chord Theorem):

(In textual terms)

In a circle, when a chord is bisected by another chord (The Bisecting Chord), twice the square of the bisecting chord is equal to the sum of the squares of the sides of the quadrilateral formed by the end points of the said two chords.

(In Geometric terms)

If BC is a chord inside a circle and D is its midpoint and ADE is another Chord bisecting BC at D, then, $2AE^2 = AB^2 + BE^2 + EC^2 + CA^2$

Given:

Chord AE bisects another chord BC at D inside the circle.

To Prove:

$2AE^2 = AB^2 + BE^2 + EC^2 + CA^2$

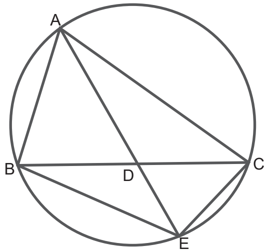


Figure : 73

Proof:

AD is median for ΔABC.

∴ As per Apollonius Theorem,

$AB^2 + AC^2 = 2(AD^2 + BD^2)$ ------- (1)

ED is the median for ΔEBC

∴ As per Apollonius Theorem,

$EB^2 + EC^2 = 2(ED^2 + BD^2)$ ---------- (2)

(1)+ (2) ⟶

$AB^2 + AC^2 + EB^2 + EC^2 = 2AD^2 + 2BD^2 + 2ED^2 + 2\,BD^2$

$= 4BD^2 + 2(AD^2 + ED^2)$

$= 4BD^2 + 2[(AD + ED)^2 - 2AD \times ED]$

$= 4BD^2 + 2[AE^2 - 2\,BD^2]$

$(\because AD \times ED = BD \times DC = BD^2)$

$= 4BD^2 + 2AE^2 - 4BD^2$

$\mathbf{2AE^2 = AB^2 + BE^2 + EC^2 + CA^2}$ **------ Proved**

*****

XIII. THE EXTENDED ANGLE BISECTOR

Theorem 21: Arc Midpoint Theorem:

In Geometric terms:

In the figure: 74, AB is an arc & E is the midpoint of arc AB. C is any point on the circle. AB &CE meet at D.

Prove $BE^2 = EC \times ED$

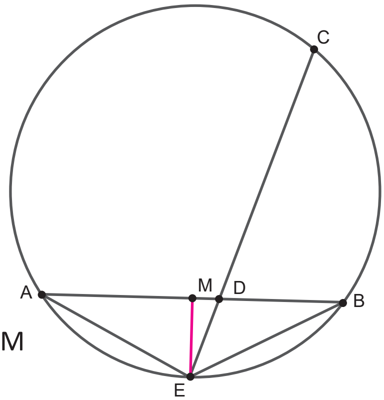


Figure : 74

Construction:

Mark the midpoint M of the chord AB. Join EM

EC x ED = ED (ED + DC)

$= ED^2 + (ED \times DC)$

$= ED^2 + (AD \times DB)$ $(\because AM = BM)$

$= ED^2 + [(BM+MD)(BM-MD)$

$= ED^2 + [BM^2 - MD^2]$

$= ED^2 + [BE^2 - EM^2 - ED^2 + EM^2]$

$BE^2 = EC \times ED$ -------------------- Hence proved.

*****

Corollary 1:

In figure :75 , AB is a chord & arc and P is the midpoint of arc AB. L, M & N are points on the circumference. PL, PM & PN (produced) intersect the chord AB at C,D & E respectively. Now,

PL x PC = PM x PD = PN x PE = PA^2

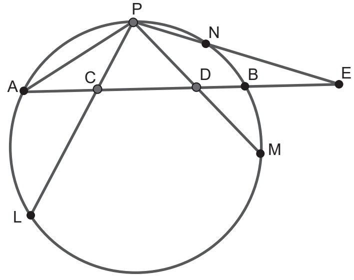


Figure : 75

Corollary 2:

In the above scenario (Figure :75), CLMD is concyclic.

*****

Theorem 22: The Extended Angle Bisector Theorem:

In Geometric terms:

In a Δ ABC, AE is the extended angle bisector of $\angle A$.

Then, $AE^2 = (AB \times AC) + BE^2$

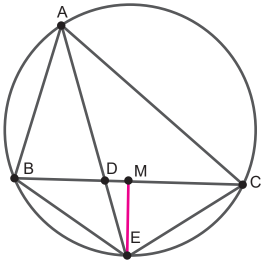


Figure : 76

Proof:

$AD^2 = (AB \times AC) - (BD \times DC)$

$= (AB \times AC) - (AD \times DE)$

Add $DE^2 + 2(AD \times DE)$ on both sides

$AD^2 + DE^2 + 2(AD \times DE) = (AB \times AC) + (AD \times DE) + DE^2$

$(AD + DE)^2 = (AB \times AC) + (AD \times DE) + DE^2$

$AE^2 = (AB \times AC) + [DE \times (AD + DE)]$

$= (AB \times AC) + [DE \times AE]$

$= (AB \times AC) + BE^2$ (by theorem 21)

$\boldsymbol{AE^2 = (AB \times AC) + BE^2}$**-------------Proved**

*****

Theorem 23 : The Sides and Angle Bisector Theorem (The SAB Theorem)

In Geometric Terms:

In a ΔABC, AE is the Extended Angle bisector of $\angle A$ meeting BC at D and the circumcircle at E. Then, AB$\times AC = AD \times AE$.

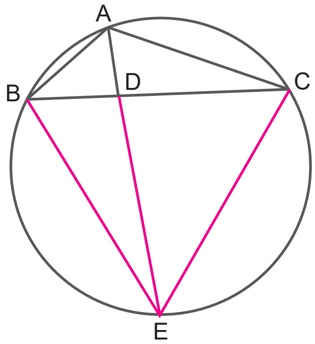


Figure : 77

Construction:

Join BE & CE

Proof:

As per the Extended Angle Bisector Theorem,

$AE^2 = (AB \times AC) + BE^2$ ------------ (1)

AE is the bisector of $\angle A$

$\Longrightarrow$E is the midpoint of the minor arc BC.

$\Longrightarrow$As per the Arc Midpoint Theorem (Proved above in page no:65)

$BE^2 = AE \times DE$ ------------------- (2)

Substituting (2) in (1), we get,

$AE^2 = (AB \times AC) + (AE \times DE)$

$\Longrightarrow AB \times AC = AE(AE - DE)$

$\Longrightarrow$ AE^2 = $AE \times AD$ ---------------------------- **Proved**

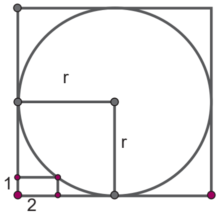
*****

XIV. AUTHOR'S SOLUTION FOR CHALLENGING PROBLEMS

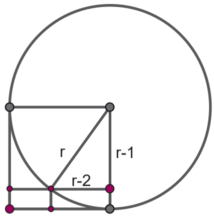
CIRCLES

Problem 1

The radius of the circle is?



Solution : 1



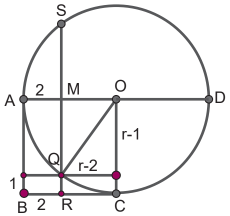
$$r^2 = (r-1)^2 + (r-2)^2$$
$$r^2 = 2r^2 - 6r + 5$$
$$r^2 - 6r + 5 = 0$$

r = 5 or 1

r = 5

****

Solution : 2



AB= BC = CO = OA = r

OM = r - 2

MQ=r-1

MD = 2r-2

AD & SQ are two chords cutting inside the circle

MQ x MS = AM x MD

(r-1) x (r-1) = 2(2) (r-1)

r-1=4(as r>1)

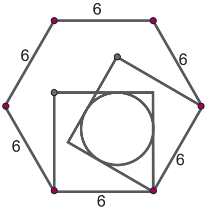
r =5

*****

Problem 2

If the side of the regular hexagon is 6 cm,

find the radius of the circle.



Solution:

Construction :

Mark the centre of the circle O. Let the point of tangency of the circle with BC & AB be P & Q respectively.

Join OP, OQ & OA.

Solution :

There are two squares ABCD & AEGH. The angle of the hexagon = 120°

The two squares are having right angles at A. Common over lapping angle between them is.

$\angle HAB = 180° - 120° = 60°$

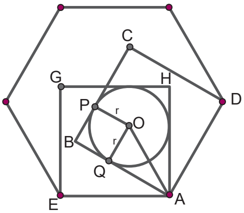
$\angle OAQ = 30°$

OP=OQ = r

AQ=6-r

$\tan 30° = \frac{1}{\sqrt{3}} = \frac{r}{6-r}$

$r = 3(\sqrt{3} - 1)$.



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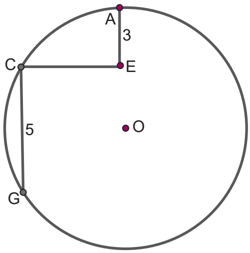
Problem 3

Find the radius of the circle if AE = 3cm, CG = 5cm, CE = 6 cm

AE ⊥ CE & CE ⊥CG

Solution :



Given than AE = 3cm,

CE = 6cm, CG=5cm

Construction:

Produce AE to meet the circle at B.

Produce CE to meet the circle at D.

Now, AB & CD are two chords intersecting at 'E'.

Draw perpendiculars from the centre 'O' to the chords CG & CD and let it meet the chords at I & H respectively.

Produce IO to meet AB at F.

Now, CI=HO=EF=2.5

AF=BF=5.5

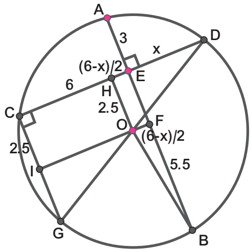
Let ED = x

As, AB & CD are two chords

intersecting at E, AE x EB = CE x ED

3x(2.5+5.5) = 6x

x=4



GD = $\sqrt{CG^2 + CD^2}$ = $\sqrt{5^2 + 10^2}$ =$\sqrt{125}$ = $5\sqrt{5}$

Radius = $\frac{5\sqrt{5}}{2} cm$

*****

Problem 4

ΔAPB is a right angled triangle & O is the circumcentre of the triangle and a tangent drawn at P. From A & B perpendiculars are drawn to meet the tangent at D & C respectively.

Prove that AD+BC = AB.

Solution : 1

$\angle APD = \angle ABP$ (Tangent chord Theorem)

$\therefore \Delta ADP \sim \Delta APB$

$$\frac{AD}{AP} = \frac{AP}{AB}$$

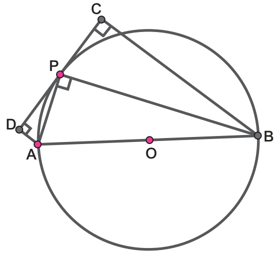
ie. $AP^2 = AB(AD)$ --------------------- (1)

Similarly, $\Delta PBC \sim \Delta ABP$

$BP^2 = AB(BC)$ ------------------------(2)

(1) & (2) → $AP^2 + BP^2 = AB(AD + BC)$ =AB^2

$\therefore AD + BC = AB$



*****

Problem 5

ABC is an acute angled triangle and O is its cirucumcentre. A circle through the points A, O and B is drawn and CA produced and CB meets the circle at P,Q respectively. Prove that CO & PQ are perpendicular to each other.

Solution : 1

Construction :

Produce CO to meet PQ at D and circle through ΔABC at E. Join EB.

Solution :

$\angle PAB = \angle PQB$ ----------------(1) (same segment)

So, $\angle CAB = \angle CQP$ -------------(2) (supplementary angles of 1)

But, $\angle CAB = \angle CEB$ ------------(3)

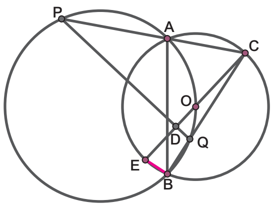
(2) & (3) →

$\angle DQC = \angle E$

So, DEBQ is concyclic.

$\angle EBQ = 90°$ (CE is diameter)

$\angle D = 90°$



*****

Problem 6

ABCD is a trapezium inscribed in a circle centered at O. It is given that AB∥CD, $\angle COD = 3\angle AOB$ and $\frac{AB}{CD} = \frac{2}{5}$. Find the ratio $\frac{area\ of\ \Delta BOC}{area\ of\ \Delta AOB}$.

Solution : 1

Let $\angle AOB = 2x$,

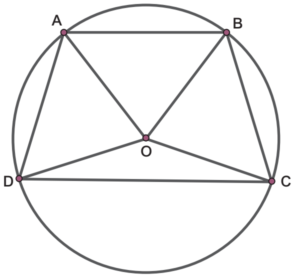
$\therefore \angle COD = 6x$

As angle subtended by chord AB in the remaining circumference is x

$\therefore AB = 2\,R\,sinx$ & CD = 2R $sin3x$

Where R is the radius,

$\frac{CD}{AB} = \frac{5}{2} = \frac{2Rsin3x}{2Rsinx} = 3 - 4Sin^2x$



$= 1 + 2[1 - 2Sin^2x]$

$= 1 + 2Cos2x = \frac{5}{2}$

$\therefore Cos2x = \frac{3}{4}$ ---------------------------(1)

Let $\angle AOD = \angle BOC = y$

$\therefore 2y + 8x = 360°$

$y = 180° - 4x$

$Siny = Sin4x$

$\therefore \frac{Area\ of\ \Delta BOC}{Area\ of\ \Delta AOB} = \frac{\frac{1}{2}R^2 Sin4x}{\frac{1}{2}R^2 Sin2x}$

$= \frac{Sin4x}{Sin2x} = \frac{2Sin2xCos2x}{Sin2x} = 2Cos2x = 2\left(\frac{3}{4}\right) = \frac{3}{2}$

*****

TRIANGLES

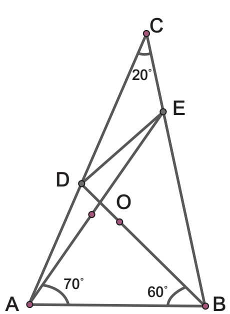
Problem 7

In ΔABC, AC=BC and $\angle C = 20°$. If D & E are points on AC & BC such that $\angle EAB = 70°$ & $\angle DBA = 60°$, then find $\angle DEA$.

For this problem, the author has given two different solutions one using Geometry and the other using trigonometry.

Solutions:

Solution : 1



In $\Delta\ CDB$,

$\frac{CD}{sin20°} = \frac{BC}{sin140°} = \frac{BC}{sin40°}$

$\therefore CD = \frac{BC}{sin40°} \times sin20°$

$= \frac{BC}{2cos20°}$ ---------------- 1

InΔ CAE, $\frac{CE}{sin10°} = \frac{AC}{sin150°} = \frac{AC}{sin30°}$ = 2AC

∴ CE = 2AC (sin10°) ------------ 2

1 & 2 →

$$\frac{CD}{CE} = \frac{\frac{BC}{2cos20°}}{2\text{AC sin}10°}$$

$$=\frac{1}{4sin10°cos20°}$$

$$= \frac{cos10°}{4sin10°\, cos10°\, cos20°} = \frac{cos10°}{sin40°} = \frac{sin80°}{sin40°}$$

$$=2cos40°$$ ---3

In Δ OBE, $\frac{BE}{BO} = \frac{sin130°}{sin30°} = \frac{sin50°}{sin30°}$ = 2cos40° ------------------ 4

3 & 4 →

In Δ CDE & Δ BEO, $\frac{CD}{CE} = \frac{BE}{BO}$ & ∠DCE = $\angle EBO = 20°$

∴ As per SAS principle,

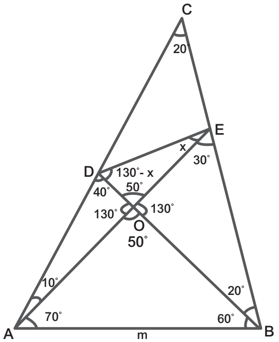
Δ CDE ||| ΔBEO.

∴ $\angle CED = \angle BOE = 130°$

∴ $\angle DEA = 20°$

*****

Solution - 2



Let BD and AE intersect at O. Let$\angle DEO = x°$

Now, from the given data, we get the following.

$\angle C$ =20°;

$\angle AOB = 50° = \angle DOE$

$\angle AOD = 130° = \angle BOE$

$\angle ADB = 40°$; $\angle AEB = 30°$

$\angle EDO = (130° - x)$; $\angle DEC = (150° - x)$

Let AB = m

Proof:

In ΔABD, $\frac{BD}{sin\,80°} = \frac{m}{sin\,40°}$

BD = $\frac{m\, sin\,80°}{sin\,40°}$ -------------------------(1)

In ΔABE, $\frac{BE}{\sin 70^\circ} = \frac{m}{\sin 30^\circ}$

BE =$\frac{m \sin 70^\circ}{\sin 30^\circ}$

BE = 2m(sin70◦) --------------------- (2)

(1) & (2) →

$\frac{BE}{BD} = \frac{2m\sin70^\circ}{\frac{m\sin80^\circ}{\sin40^\circ}}$ = $\frac{2 \sin 70^\circ \sin 40^\circ}{\sin 80^\circ}$

$\frac{BE}{BD} = \frac{2 \sin 70^\circ \sin 40^\circ}{2 \sin 40^\circ \cos 40^\circ}$ =$\frac{\sin 70^\circ}{\cos 40^\circ}$

$\frac{BE}{BD} = \frac{\sin 70^\circ}{\sin 50^\circ}$ --------------------------- (3)

In ΔBED, $\frac{BE}{BD} = \frac{\sin(130^\circ - x)}{\sin(30^\circ + x)}$ --------- (4)

(3)& (4) →

$\frac{\sin(130^\circ - x)}{\sin(30^\circ + x)}$=$\frac{\sin 70^\circ}{\sin 50^\circ}$=$\frac{\sin 110^\circ}{\sin 50^\circ}$

$\frac{\sin(130^\circ - x)}{\sin(30^\circ + x)}$=$\frac{\sin 110^\circ}{\sin 50^\circ}$

If we put x = 20, the above equation is satisfied.

ie LHS = RHS only if x = 20°.

$\frac{\sin(130^\circ - 20^\circ)}{\sin(30^\circ + 20^\circ)}$= $\frac{\sin 110^\circ}{\sin 50^\circ}$

Cross multiplying,

Sin50° Sin(130° – x) = Sin110° Sin(30° +x)

2Sin(130° – x) Sin50° = 2Sin110° Sin(30° +x)

2SinA SinB = Cos(A - B) – Cos(A + B)

ie Cos(80° –x) – Cos(180° – x) = Cos(80° – x) - Cos(140° + x)

ie 180° - x = 140° + x

ie 2x = 40° & x = 20°

*****

Problem 8

Given: A, B & C are non collinear points. Segment AB is trisected by points D & E, and F is the mid-point of segment AC. DF & BF intersect CE at G & H respectively. If area of Δ DEG is 18 Sq.units, find the area of Δ FGH.

Solution:

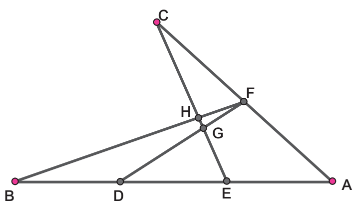
$$\frac{FC}{CA} = \frac{1}{2}; \frac{AF}{FC} = 1; \frac{ED}{DA} = \frac{1}{2}; \frac{EB}{BA} = \frac{2}{3}; \frac{AE}{ED} = 1$$

As per Menelaus Theorem,

$$\frac{AF}{FC} \times \frac{CG}{GE} \times \frac{ED}{DA} = 1$$

$\therefore CG = 2GE$ -------------------- (1)

Again, as per Menelaus Theorem,



$$\frac{AF}{FC} \times \frac{CH}{HE} \times \frac{EB}{BA} = 1$$

$\therefore \frac{CH}{HE} = \frac{3}{2}$

CH = $\frac{3}{2}$ HE --------------------(2)

(1)– (2) → CG- CH = 2GE - $\frac{3}{2}$ HE

ie GH = 2GE – [$\frac{3}{2} GE + \frac{3}{2}$GH]

= $\frac{1}{2} GE - \frac{3}{2}$GH

Ie $\frac{5}{2} GH = \frac{1}{2} GE$

5GH = GE --------------------------- (3)

As per Menelaus Theorem,

$$\frac{AE}{ED} \times \frac{DG}{GF} \times \frac{FC}{CA} = 1$$

$$\frac{DG}{GF} = 2$$

DG = 2 GF --------------------------- (4)

[HGF] = $\frac{1}{2} \times Sin\angle HGF\ (HG)(GF)$ ------------ (5)

[DGE] = $\frac{1}{2} \times Sin\angle DGE\ (GE)(DG)$ ------------ (6)

(3), (4), (5) & (6) →

$\therefore [DGE] = 10\ [HGF]$

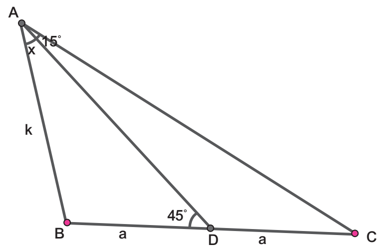
$[HGF] = \frac{18}{10} = 1.8\ sq.units.$

*****

Problem 9

In ΔABC, AD is a median from A to BC such that $\angle DAC = 15°$ and$\angle ADB = 45°$, find x.

Solution :



$\angle ACD = 45° - 15° = 30°$

$$\frac{k}{Sin45°} = \frac{a}{Sinx}$$

$k = a(\frac{Sin45°}{Sinx})$ --------------------------------(1)

$$\frac{k}{Sin30°} = \frac{2a}{Sin(x+15°)}$$

2k = $\frac{2a}{Sin(x+15°)}$

k=$\frac{a}{Sin(x+15°)}$ --------------------------- (2)

(1) & (2)→

$$\frac{Sin45°}{Sinx} = \frac{1}{Sin(x+15°)}$$

$$\frac{Sinx}{Sin(x+15°)} = \frac{1}{\sqrt{2}}$$

$Sinx = Sin\ (x+15°)Cos45°$

$= \dfrac{Sin(60° + x) + Sin(30° - x)}{2}$

$2Sinx = Sin60°Cosx + Cos60°Sinx + Sin30°Cosx - Cos30°Sinx$

$Sinx(2 - Cos60° + Cos30°) = Cosx(Sin60° + Sin30°)$

$Sinx\left(2 - \frac{1}{2} + \frac{\sqrt{3}}{2}\right) = Cosx(\frac{\sqrt{3}}{2} + \frac{1}{2})$

tanx $= \dfrac{\frac{1+\sqrt{3}}{2}}{\sqrt{3}\,(\frac{1+\sqrt{3}}{2})} = \dfrac{1}{\sqrt{3}}$

x=30°

*****

Corollary 1:

If AD is a median from A to BC in such that $\angle ADB = 45°$ and BE is drawn perpendicular to AD meeting AC at F, then, prove that tanθ $= \frac{AF}{AC}$.

Proof:

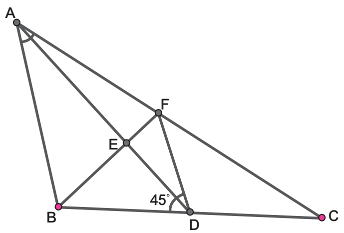
In ΔBED,

$\angle BED = 90°$ & $\angle EDB = 45°$ →

$\angle EBD = 45°$

∴BE = ED

In$\Delta DEF, \angle DEF = 90°$



tanθ= $\frac{EF}{DE} = \frac{EF}{BE}$ ----------------------(1)

(As BE = DE)

In ΔFBC, AED is a transversal

By Menelaus Theorem,

$\frac{CA}{AF} \times \frac{FE}{EB} \times \frac{BD}{DC} = 1$

As, BD = DC

$\frac{FE}{EB} = \frac{AF}{CA}$ ---(2)

(1) & (2) gives tan$\theta = \frac{AF}{AC}$.

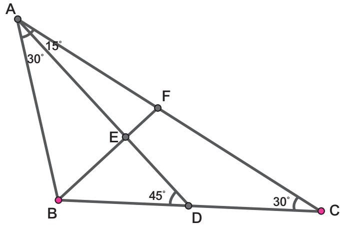
*****

Corollary 2

If AD is a median from A to BC such that $\angle ADB = 45°$ and $\angle DAC = 15°$. BE is drawn perpendicular to AD meeting AC at F, then $AD \times BF = 2BD^2$.

Proof:

We have already proved that $\angle BAD = 30°$ & $\angle DCA = 30°$



$\angle ACB = \angle DAB = 30°$

$\angle CBF = \angle ADB = 45°$

$\Delta ABD \sim \Delta CFB$

$$\frac{BD}{FB} = \frac{AD}{BC}$$

$AD \times BF = BD \times BC$

$AD \times BF = BD \times 2BD$ (As $BC = 2BD$)

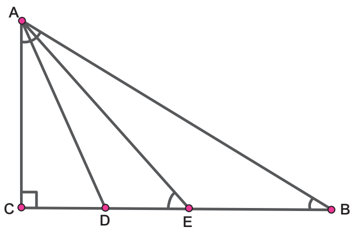
$AD \times BF = 2BD^2$

*****

Problem 10

ABC is a triangle right angled at C. BC is divided by points D & E into three equal parts. Find the sum of the angles AEC, ADC & ABC if BC = 3AC.

Solution :1



Let AC=a

$\therefore BC = 3AC = 3a$

$\Longrightarrow AD = DE = EC = a$

In ΔADC, as $AC = DC$

and $\angle ACD = 90°$, $\angle ADC = 45°$

$\tan \angle AEC = \frac{AC}{EC} = \frac{a}{2a} = \frac{1}{2}$

$\tan \angle ABC = \frac{AC}{BC} = \frac{a}{3a} = \frac{1}{3}$

$$\tan(\angle AEC + \angle ABC) = \frac{\tan\angle AEC + \tan\angle ABC}{1 - \tan\angle AEC \tan\angle ABC} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$$

$\angle AEC + \angle ABC = 45°$

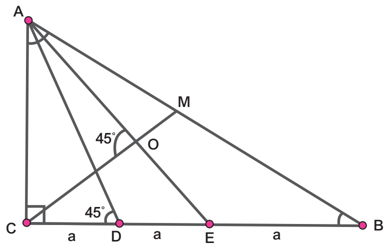
$\angle ADC + \angle AEC + \angle ABC = 90°$

*****

Solution :2

Let M be the midpoint of AB.

Let CM and AE intersect at O.



Let AC = CD = DE = EB = a

$\therefore \angle ADC = 45°$

$\therefore AE = \sqrt{4a^2 + a^2} = a\sqrt{5}$ ----------(1)

For $\Delta ABE, MOC$ is a transversal,

$\therefore$ As per Menelaus Theorem,

$$\frac{BM}{MA} \times \frac{AO}{OE} \times \frac{CE}{CB} = 1$$

ie $\frac{1}{1} \times \frac{AO}{OE} \times \frac{2}{3} = 1$

ie $\frac{AO}{OE} = \frac{3}{2}$ --------------------------------(2)

(1) & (2) → OE = $a\sqrt{5} \times \frac{2}{5} = \frac{2a}{\sqrt{5}}$ ------------ (3)

(1) & (3) → $AEXEO = a\sqrt{5} \times \frac{2a}{\sqrt{5}} = 2a^2$ ------------(4)

$CE \times ED = 2a \times a = 2a^2$ ---------------------(5)

(4) & (5) → $AE \times EO = CE \times ED$

$\therefore$ $AODC$ is concyclic. [AO & CE are chords intersecting outside the circle at E.]

$\therefore \angle AOC = \angle ADC = 45°$

ie $\angle AOC = \angle AEC + \angle OCE = \angle AEC + \angle ABC = 45°$

$\therefore \angle ADC + \angle AEC + \angle ABC = 90°$

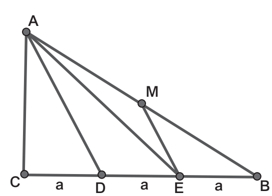
*****

Solution :3

Let AC = CD = DE = EB = a, Let M be the midpoint of AB

$\therefore AE = \sqrt{5}a, \quad AB = \sqrt{10}\, a, \;\; AM = \frac{\sqrt{10}\, a}{2} = \frac{\sqrt{5}\, a}{\sqrt{2}}$

$$\frac{AE}{AM} = \frac{\sqrt{5}\, a}{\sqrt{5}a/\sqrt{2}} = \sqrt{2}$$ --------------------(1)



$\frac{AB}{AE} = \frac{\sqrt{10}\, a}{\sqrt{5}a} = \sqrt{2}$ ----------------------------(2)

In ΔAEB & $\Delta AME, \angle BAE = \angle EAM$ -----(3)

From (1) & (2) $\frac{AB}{AE} = \frac{AE}{AM}$ ---------------------- (4)

From (3) & (4), By SAS principle $\Delta AEB \approx \Delta AME$

$\therefore \angle EBA = \angle MEA$ ----------------------------(5)

But $\angle MEA = \angle DAE$

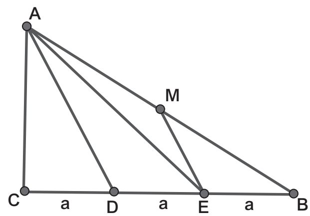
Since M and E are mid points of AB and DB respectively. AD ∥ ME.

Implies $\angle AEC + \angle ABC = \angle AEC + \angle DAE = 45°$

$\angle ADC + \angle AEC + \angle ABC = 90°$

*****

Solution :4



Let M be the midpoint of AB. Join ME

$\angle ADC = 45° = \angle AED + \angle EAD$

It's enough to prove that $\angle EAD = \angle ABE$

Let AC= CD=DE=EB=a, AD = $\sqrt{2}a$

$\therefore EA = \sqrt{5}$a & MB = $\frac{\sqrt{10}a}{2} = \frac{\sqrt{5}a}{\sqrt{2}}$

In ΔADE & ΔBEM

Since M and E are midpoints of AB and DB respectively. AD ∥ ME

$$ME = \frac{1}{2}AD = \frac{a}{\sqrt{2}}$$

$$\angle AED = \angle BEM, \frac{AD}{EB} = \sqrt{2}, \frac{DE}{EM} = \sqrt{2}$$

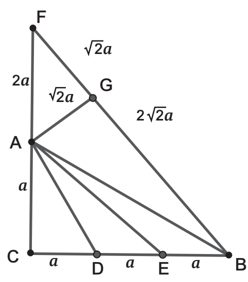
By SAS principle $\Delta ADE \sim \Delta BEM$

$\therefore \angle ABE = \angle EAD$

$\angle ADC + \angle AEC + \angle ABC = 90°$

*****

Solution :5



Let AC = CD = DE = EB = a

Produce CA to F such that AF=2a.

Join BF. Draw AG ⊥ FB.

Now, ΔFCB & ΔFAG are isosceles,

$\angle CFB = \angle CBF = 45°$

FG=AG=$\sqrt{2}a$, $GB = BF - FG = 3\sqrt{2}a - \sqrt{2}a$ = $2\sqrt{2}a$

Now, ΔACE & ΔAGB, $\frac{AC}{CE} = \frac{1}{2} = \frac{AG}{GB}, \angle C = \angle G = 90°$

$\therefore \Delta ACE \sim \Delta AGB \Longrightarrow \angle AEC = \angle ABG$

$\angle AEC + \angle ABC = \angle ABG + \angle ABC = 45°$

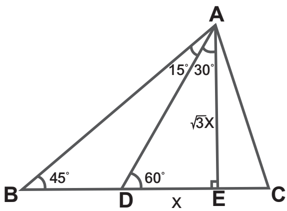
$\angle ADC + \angle AEC + \angle ABC = 90°$

*****

Problem 11

In, ΔABC, ∠ABC = 45°, AD is a cevian meeting BC at D such that DC = 2 BD and ∠ADC = 60°, find ∠ACB.

Solution : 1



Draw AE ⊥ BC

Let DE be x

Δ *ADE is a special right* Δ *with angles* $30°, 60° \ \& \ 90°$

$\therefore AE = \sqrt{3}\,x = BE$

$\therefore BD = (\sqrt{3}x - x) \ \& \ EC = 2(\sqrt{3}\,x - x) - x$

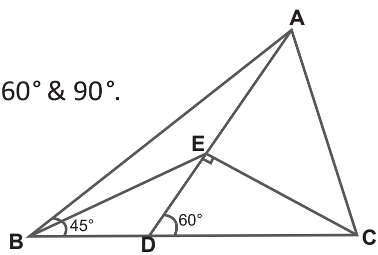
EC = $2\sqrt{3}x - 3x$

tan C = $\frac{\sqrt{3}\,x}{2\sqrt{3}x - 3\,x}$ = $2+\sqrt{3}$ $=$ tan 75°

$\therefore C = 75°$

*****

Solution :2



Draw CE ⊥ AD. Join BE.

Now, Δ CDE is a right Δ $with$ angles 30°, 60° & 90°.

$\therefore DC = 2DE = 2BD \ (given)$

$\therefore DE = DB$

$\therefore \ \angle EBD = \frac{180° - 120°}{2}$ = 30°

$\therefore EB = EC$ -------------------------------(1)

$\therefore \angle EBA$ = 15° $= \angle EAB$ $(\angle EBA = \angle ADC - \angle ABD = 60° - 45°)$

$\therefore EA = EB$ ------------------------------- (2)

(1) & (2) --> EA = EB = EC

$\therefore \angle ACB = 75°$

*****

Solution : 3

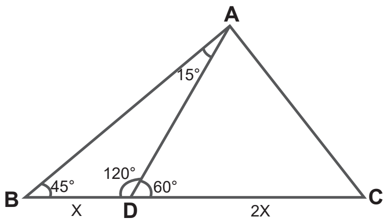
Proof : Let $BD = x$, $\therefore DC = 2x$.

$\therefore BC \times DC = 6x^2$ ------------------------- (1)

As $\angle ADC = 60°$, $\angle BAD = 15°$ & $\angle ADB = 120°$

By Sine formula,

$$\frac{AB}{sin120°} = \frac{BD}{sin15°}$$



$$AB = \frac{x \times sin120°}{sin15°} = \frac{\frac{\sqrt{3}x}{2}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \left(\frac{\sqrt{6}}{\sqrt{3}-1}\right)x$$

$$\therefore AB = \left(\frac{\sqrt{6}}{\sqrt{3}-1}\right)x$$

By Cosine formula,

$$AC^2 = AB^2 + BC^2 - 2(AB)(BC)\,cos45°$$

$$= \frac{6}{(\sqrt{3}-1)^2}x^2 + 9x^2 - 2\left(\frac{\sqrt{6}}{\sqrt{3}-1}\right)x \times (3x)\frac{1}{\sqrt{2}}$$

$$= \frac{6x^2+9x^2(\sqrt{3}-1)^2 - 6\sqrt{3}(\sqrt{3}-1)x^2}{(\sqrt{3}-1)^2}$$

$$= \frac{6x^2 + 9x^2(4 - 2\sqrt{3}) - 18x^2 + 6\sqrt{3}x^2}{4 - 2\sqrt{3}}$$

$$= \frac{(6+36-18\sqrt{3}-18+6\sqrt{3})x^2}{4-2\sqrt{3}}$$

$$= \frac{(24-12\sqrt{3})x^2}{4-2\sqrt{3}} = \frac{6(4-2\sqrt{3})}{4-2\sqrt{3}}x^2 = 6x^2$$

$\therefore AC^2 = 6x^2$ ------------------------- (2)

By (1) & (2) , $AC^2 = BC \times DC$

$$\therefore \frac{AC}{DC} = \frac{BC}{AC}$$

$\therefore$ *In* ΔABC & ΔDAC,

i) $\frac{AC}{DC} = \frac{BC}{AC}$

i) $\angle C$ *is common*

$\therefore$ *By SAS similarity criterion,* $\Delta\, ABC \sim \Delta DAC$

$\angle DAC = \angle ABC = 45° \implies \angle ACD = 75°$

$\therefore \angle ACB = 75°$

*****

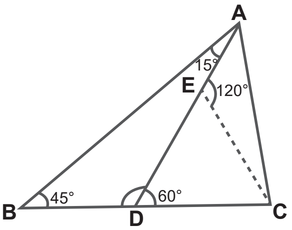
Solution : 4

Choose a point on AD such that $\angle DCE = 60°$.

$\therefore\ \angle DCE = 60°. \therefore \Delta DCE$ *is equilateral* .

Let BD = x. $\therefore DC = CE = ED = 2x$

Now,



$$\frac{AD}{BD} = \frac{sin45°}{sin15°} = 3 - 4\, sin^2 15°$$

= $1+2\cos 30°$

= $1+\sqrt{3}$

AD= $(1+\sqrt{3})\ x$

$\therefore AE = (1+\sqrt{3})\ x - 2x = (\sqrt{3}-1)\ x$

$\frac{AE}{EC} = \frac{(\sqrt{3}-1)x}{2x} = \frac{(\sqrt{3}-1)}{2}$ ---------------------------- (1)

$\frac{BD}{AD} = \frac{x}{(1+\sqrt{3})\ x} = \frac{1}{(1+\sqrt{3})} = \left(\frac{\sqrt{3}-1}{2}\right)$ -------------- (2)

$\angle AEC = \angle ADB = 120°$ ------------------------ (3)

(1), (2) & (3) --> as per SAS principle,

$\Delta\, ABD\ |||\ \Delta CAE$

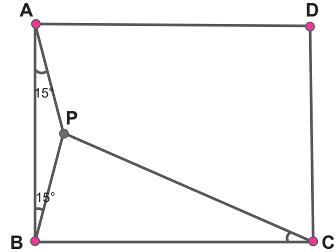
$\angle ACE = \angle BAD = 15°$

$\therefore \angle ACD = 60° + 15° = 75°$

*****

Problem 12

Find the value of θ, if ∠PAB = ∠ PBA = 15° .
The question sender was taken aback on viewing two simple solutions for such a challenging problem.



Solution :1

$\frac{BC}{BP} = \frac{AB}{BP} = \frac{Sin150°}{Sin\ 15°} = \frac{Sin30°}{Sin15°} = 2Cos15°$ ---(1)

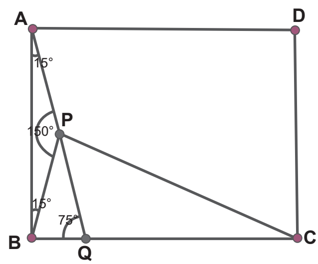
$\frac{BP}{BQ} = \frac{Sin75°}{Sin\ 30°} = 2Sin75° = 2Cos15°$ ---(2)

In ΔPBC & ΔQBP, 1 & 2 →

$\frac{BC}{BP} = \frac{BP}{BQ}$ & $\angle B$ is common (75°)

∴ ΔPBC & ΔQBP are similar

∴ $\angle BPQ = \angle BCP = 30°$



*****

Solution: 2

Construction:

Produce AP to meet BC at Q.

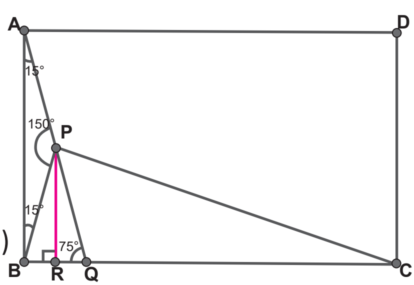
Draw PR ⊥ BC.

Now BR = RQ. Let AB be ‘a’.

$\angle PBQ$ & $\angle PQB$ are 75°

PB = PQ (sides opposing equal angles)

∴ PA = PB = PQ.



PR ∥ AB & AP = PQ

As per BPT, $PR = \frac{1}{2}\ AB = \frac{1}{2}\ a$ ---------------------------- (1)

$\frac{BR}{PR}$ = tan 15° ; BR = PR (tan 15°)

$= \frac{1}{2}$ a (tan 15°) ------------------ (2)

$\tan\theta = \frac{PR}{RC} = \frac{\frac{1}{2}a}{a - \frac{1}{2}a\,tan15°}$

$= \frac{1}{2 - tan15°}$

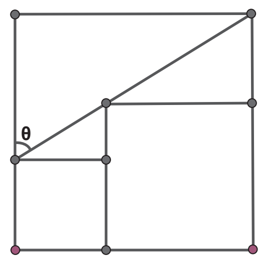
$= \frac{1}{2 - \left[\frac{\sqrt{3}-1}{\sqrt{3}+1}\right]} = \frac{1}{\sqrt{3}}$ = tan 30°

$\therefore \theta$ = 30°

*****

Problem 13

Find tanθ in the given figure



Solution:

Let $\angle EAD = x$

Let $tanx = k$

$\therefore tan\theta = \frac{1}{\tan(90° - \theta)} = \frac{1}{tanx}$ ----------(1)

$tanx = \frac{ED}{AD} = k$ --------------------------(2)

$\therefore if$ AD = a, ED= ka

And EF = CD+DE=a+ka=a(1+k)

$\angle HEF = \angle EAD = x$

$\therefore tanx = \frac{a}{a(k+1)} = \frac{1}{(k+1)}$ ----------------(3)

(2) & (3) →

$k = \frac{1}{(k+1)}$

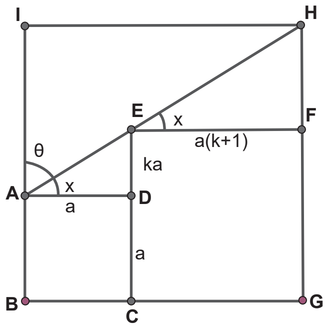
ie k (k+1) =1

ie $k^2 + k - 1 = 0$

k=$\frac{-1 \pm \sqrt{1^2+4}}{2}$

=$\frac{-1 \pm \sqrt{5}}{2}$ =$\frac{\sqrt{5}-1}{2}$

$k = \frac{\sqrt{5}-1}{2}$



ie $tan x = \frac{\sqrt{5}-1}{2}$

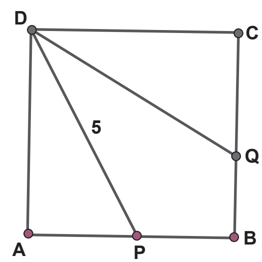
$\therefore tan\theta = \frac{2}{\sqrt{5}-1}$

*****

Problem 14

P is a point on side AB of square ABCD such that DP =5cm.
DQ is the angle bisector of $\angle CDP$ where Q is a point on side BC. Then find the length of (CQ+AP)

Solution: 1



Let $\angle APD\ be\ y$

$\therefore \angle PDC = y$

And $\angle PDQ = \angle CDQ = \frac{y}{2}$

Let AD = DC = a

Let AP = b & CQ = c

$tan\ \frac{y}{2} = \frac{c}{a}$

$\therefore tany = \frac{2\left(\frac{c}{a}\right)}{1-\frac{c^2}{a^2}} = \frac{2ac}{(a^2-c^2)}$ -------------- (1)

But in $\Delta\ ADP$,

$tan\ y = \frac{a}{b}$ ------------------------------------- (2)

(1) & (2) ⟶

$$\frac{2ac}{(a^2-c^2)} = \frac{a}{b}$$

ie $\frac{2c}{(a^2-c^2)} = \frac{1}{b}$

2bc = $a^2 - c^2 = [25 - b^2] - c^2$

ie $(b+c)^2$=25; ie b+c =5

*****

Solution 2:

$a = 5\ sin\ y$ ------------------ (1)

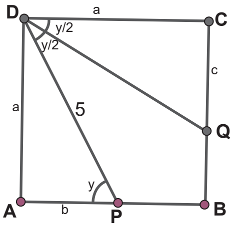
b $= 5\ cos\ y$

c $= a\ \times tan\frac{y}{2} = 5\ sin\ y\ \times tan\ \frac{y}{2}$ (from (1))

$\therefore b + c = \ 5cosy + 5\ siny\ \times tan\ \frac{y}{2}$

$= 5(1 - 2sin^2\frac{y}{2} + \ 2sin\frac{y}{2}\ cos\ \frac{y}{2}\ \times \frac{sin\frac{y}{2}}{cos\frac{y}{2}})$

= 5



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XV. AUTHOR'S CREATIONS

TRIANGLES

Problem 1

A straight line makes positive intercepts on the X & Y axis at A & B respectively. The distance between A & B is 70 units. Another straight line y = x intersects AB at C and AC: CB = 4: 3. Find the equation of the straight line AB.

Solution : 1

The slope of the straight line y = x is 1 and it passes through the origin O. Therefore the angle made by the straight line at the origin is 45°.

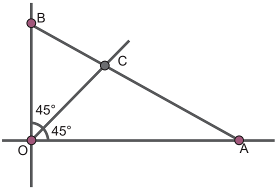
In other words, it bisects $\angle BOA$. In ΔOAB, OC is the bisector of $\angle BOA$.

As per Angle Bisector Theorem, $\frac{OA}{OB} = \frac{AC}{CB} = \frac{4}{3}$ (given)

∴3OA = 4OB.

∴ the coordinates of A & B can be assume to be A (4k, 0) and B (0, 3k).

Now, as per Distance formula,



$(4k + 0)^2 + (0 + 3k)^2 = 70^2$

$25k^2 = 4900$

K=14

∴ $xintercept$ =4k =56

Y intercept = 3k=42

The equation of the straight line AB is $\frac{x}{56} + \frac{y}{42} = 1$.

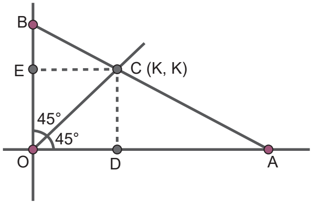
*****

Solution : 2

The coordinates of x & y will be equal throughout the line y = x.

∴ The coordinates at C can be assumed as (k,k)

OD = OE = k



As per Basic proportionality Theorem,

$\frac{BO}{OE} = \frac{BA}{Ac} = \frac{7}{4}$ (given)

$BO = \frac{7}{4}k$

Similarly $\frac{AO}{OD} = \frac{AB}{BC} = \frac{7}{3}$

AO = $\frac{7}{3}k$

∴The coordinates of A & B can be assumed as A($\frac{7}{3}k, 0$) & B(0, $\frac{7}{4}k$)

As per Distance formula, $\left(\frac{7}{3}k\right)^2 + \left(\frac{7}{4}k\right)^2 = 70^2$

$\frac{3^2\times 49k^2+4^2\times 49k^2}{3^2\times 4^2}$ =70^2

Simplifying we get k =24

∴ x intercept = $\frac{7}{3} \times 24 = 56$

y intercept = $\frac{7}{4} \times 24 = 42$

∴ The equation of the straight line AB is $\frac{x}{56} + \frac{y}{42} = 1.$

*****

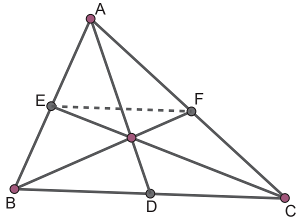
Problem 2

In ΔABC, AD is the median meeting BC at D. 'O' is a point on AD. If cevians drawn from B & C through 'O' meet AC & AB respectively at F & E then EF is parallel to BC.

(For this result, the Geometricians across the world have given their own lengthy proofs. But the author has proved this result very simply in just two steps using Ceva's theorem.)

Solution:

As the cevians AD, BF, CE are concurrent,



By Ceva's Theorem,

$\frac{AE}{EB} \times \frac{BD}{DC} \times \frac{CF}{FA} = 1$

Given that, AD is the median, so BD = DC.

$\frac{AE}{EB} \times \frac{1}{1} \times \frac{CF}{FA} = 1$

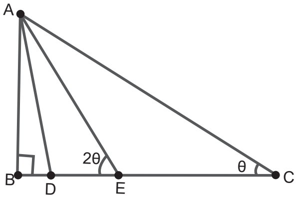
$\frac{AE}{EB} \times \frac{CF}{FA} = 1$ gives $\frac{AE}{EB} = \frac{AF}{CF}$

By converse of BPT, EF is parallel to BC.

*****

Problem 3

In the picture, ΔABC is right angled at B. D & E are points on BC such that AE is the bisector of $\angle DAC$ & $\angle AED = 2\angle C$. If DE:EC= $1:(1+\sqrt{2})$.
Prove that $AB^2 = BD \times BC$.



Solution : 1

$\angle DAE = \angle EAC = 2\theta - \theta = \theta$

$\angle ADE = 180 - 3\theta$

$\frac{CE}{ED} = \frac{AC}{AD} = \frac{(1+\sqrt{2})}{2}$ ------ (1) [Angle bisector theorem]

$\frac{AC}{AD} = \frac{Sin(180-3\theta)}{Sin\theta}$ [Sin rule]

$= \frac{Sin3\theta}{Sin\theta} = 3 - 4Sin^2\theta = 1 + 2Cos2\,\theta$ ---------(2)

(1)& (2) → $1 + 2Cos2\,\theta = 1+\sqrt{2}$

$\therefore Cos2\theta = \frac{1}{\sqrt{2}}$

$\theta = 22.5$

$\therefore \angle BAD = 90° - 3\theta = 22.5°$

In ΔABC & ΔDBA

$\angle ABC = \angle DBA = 90°$

$\angle BCA = \angle BAD = 22.5°$

By, AA similarity Criterion, $\Delta ABC \sim \Delta DBA$

$\frac{AB}{BC} = \frac{BD}{AB}$ gives $AB^2 = BD \times BC$.

*****

Problem 4

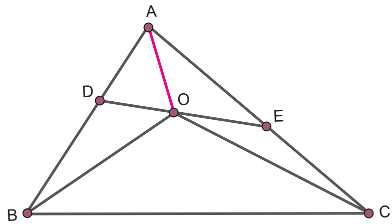
In a ΔABC, the bisectors of $\angle B$ &$\angle C$ meet at O.

DE is a straight line drawn through O such that AD= AE.

Prove that $DE^2 = 4(BD \times CE)$.

Solution : 1

Construction: Join AO



In ΔABO, AO is the angle bisector of $\angle A$,

$$\angle AOB = 180° - \frac{(A+B)}{2}$$

$$= 180° - \frac{(180°-C)}{2}$$

$$= 90° + \frac{C}{2}$$

AO is the perpendicular bisector ΔADE.

OD = OE & $\angle AOD = 90° \rightarrow$

$$\angle DOB = \frac{C}{2},$$

Similarly $\angle EOC = \frac{B}{2}$.

In $\Delta BDO \ and \ \Delta OEC,$

$\angle DOB = \frac{C}{2} = \angle OCE, \ \angle EOC = \frac{B}{2} = \angle OBD$, (AA similarity)

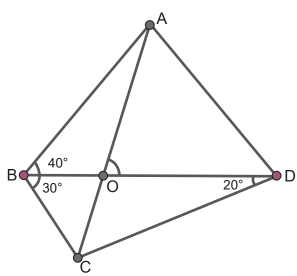
$$\Delta BOD \approx \Delta OCE, \frac{BD}{OE} = \frac{DO}{EC} = \frac{OB}{OC}$$

OD $\times OE = BD \times EC \rightarrow OD^2 = BD \times EC \rightarrow DE^2 = 4BD \times EC$

*****

QUADRILATERALS

Problem 5



In quadrilateral ABCD, AC & BD meet at O; AB = AD;

∠ABD = 40°; ∠CBD = 30°; and ∠BDC = 20°.

Find the measurement of ∠AOD.

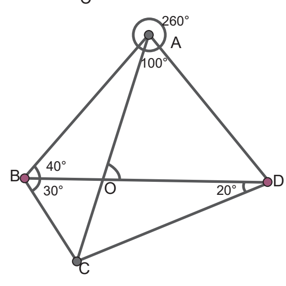
Solution:

AB = AD

$\angle BAD = 100°$ & $\angle BCD = 130°$

∴ Concave $\angle BAD = 360°$ -100 = 260° = $2\angle BCD$

ie. if a circle is drawn with centre A & radius AB, it will pass through C.



∴ AB = AC (radius)

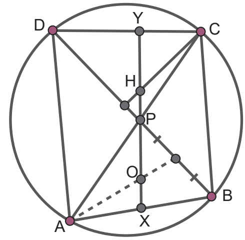
∴ $\angle ACB = \angle ABC = 70°$

And $\angle BOC = \angle AOD = 80°$

*****

Problem 6

The diagonals AC and BD of a cyclic quadrilateral ABCD intersect at P. Let O be the centroid of Δ APB and H be the orthocentre of Δ CPD. Show that the points H,P,O are collinear.



Solution:

Draw the median PX (from vertex P) for Δ APB. Produce XP to meet DC at Y. Now as per Brahmagupta's Theorem, PY will be perpendicular to DC and be the altitude from Vertex P for Δ CDP.

Since, PX is the median of ΔAPB, the centroid 'O' of ΔAPB will lie on PX only. ------------(1)

Since, PX is the altitude of ΔAPB, its orthocentre H will

lie on PX only. ------------------ (2)

(1) &(2) → **P, O and H will be collinear on the straight line XPY.**

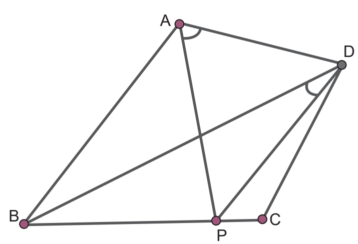
*****

Problem 7

Given a quadrilateral ABCD where

BD bisects ∠B , P is a point on BC such that PD bisects ∠APC.

Show that ∠BDP + ∠PAD = 90°



Construction :

Draw the angle bisector of $\angle BAP$ & let is meet BD at O. Join PO.

Solution :

In ΔABP,

AO & BO are angle bisectors.

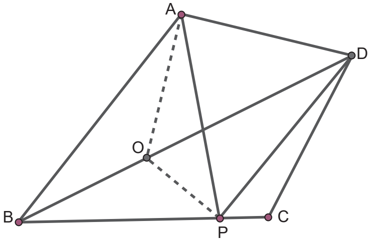
$\therefore$ $PO$is also angle bisector.

$\therefore \angle OPD = 90°$ ------------------------ (1)

$\angle APC = \angle BAP + \angle ABP \quad [\angle A + \angle B]$

$\therefore \angle APD = \frac{(\angle A+\angle B)}{2}$ -------------------(2)

$\therefore \angle AOD = \frac{(\angle A+\angle B)}{2}$ ----------------- (3)



(2) & (3) → $\angle APD = \angle AOD$

→ $AOPD$ is concyclic

$\therefore \angle POD = \angle PAD$ ---------------------- (4)

(1) → $\angle POD + \angle BDP = 90°$ ---------- (5)

(4) & (5) → $\angle BDP + \angle PAD = 90°$ ----------------------------- Proved

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XVI. AUTHOR'S ARTICLES

AN ESSAY ON PYTHAGORAS THEOREM

The Pythagoras theorem explains the relationship that exists among the three sides of a right triangle. Crisply, the theorem can be stated as below:

If 'a' is the smallest side of the right Δ, 'b' is the next bigger side of the same Δ and 'c' is the hypotenuse of the same Δ, then, $c^2 = a^2 + b^2$.

To explain this theorem to the students, the teachers are using the combination (3, 4, & 5) as the measurements for the sides, where $3^2 + 4^2 = 5^2$. The teachers are also using the multiples of this combination such as (6,8,10) (9,12,15) and so on.

Some teachers are also using another combination (5,12,13) where $5^2 + 12^2 = 13^2$. They also use the multiples of this combination (10,24,26), (15,36,39) and so on.

Except the above two combinations viz. (3,4,5) and (5,12,13) and their multiples, we normally do not come across teachers who are using other available combinations and their multiples. In fact, there are infinity number of such combinations.

Look at the following table.

a	1	3	5	7	9	11	13	15	17……and so on
b	0	4	12	24	40	60	84	112	144….and so on
c	1	5	13	25	41	61	85	113	145….and so on

In the above table, every combination (column) represents the property $c^2 = a^2 + b^2$. Excepting the 1st combination (1, 0, 1), all the other combinations and their multiples can be used to explain the Pythagoras theorem. The first row of the above table represents the smallest side 'a' and the 2nd row represents the next bigger side 'b' and the 3rd row of the table represents the hypotenuse 'c'.

Interestingly, the numbers in the first row represent an Arithmetic Progression (AP) with a mean-difference of 2. The 2nd row 'b' also represents a regular series, where the difference between the adjacent numbers represents an AP with a mean-difference of 4. The 3rd row 'c' also represents a progression where the difference between the adjacent numbers represents an AP with a mean-difference of 4.

The teachers can also use these combinations and their multiples to explain Pythagoras theorem.

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PYTHAGORAS THEOREM VS BHODHAYAN THEOREM

The Pythagoras theorem in a nutshell is as follows. If a and b are the right angle-making sides of a right triangle,

Then, 'c' the hypotenuse =$\sqrt{a^2+b^2}$

$ie \quad c^2 = a^2 + b^2$

The Bhodhayan poem claims that if *a* & *b* are the bigger and smaller right angle-making sides of a right triangle,

Then, 'c' the hypotenuse = $\frac{7}{8}a + \frac{1}{2}b$

The Pythagoras theorem is a proved one and there is a solid Geometrical proof for the same. But Bhodhayan poem is not a proved one. Only certain examples are given in support of the same. Examples cannot constitute proof.

Now, let us study hereunder the relationship between the Pythagoras theorem and Bhodhayan poem (Bhodhayan concept)

Let '*a*' be the bigger side and '*ka*' be the smaller side forming the right triangle, where k is <1.

Now as per Pythagoras theorem,

The length of the hypotenuse = $\sqrt{a^2+(ka)^2}$

$= \sqrt{a^2(1+k^2)}$ ------------ (1)

But as per Bhodhayan concept,

length of the hypotenuse = $\left(\frac{7}{8}a + \frac{1}{2}ka\right)$

$$= \left(\frac{7a+4ka}{8}\right)$$

$$= \frac{a[7+4k]}{8} \qquad \text{------------------(2)}$$

Certainly $(1) \neq (2)$

But in certain cases, the result obtained through Bhodhayan concept tallies with the Pythagoras result. Under what circumstances? Let us analyse:

In those situations, (1) = (2)

ie $\sqrt{a^2(1+k^2)} = \frac{a(7+4k)}{8}$

ie $a\sqrt{(1+k^2)} = \frac{a(7+4k)}{8}$

ie $(1+k^2) = \frac{(4k+7)^2}{64}$

ie $64k^2 + 64 = 16k^2 + 56k + 49$

ie $48k^2 - 56k + 15 = 0$

ie $48k^2 - 36k - 20k + 15 = 0$

ie $12k(4k-3) - 5(4k-3) = 0$

ie $(4k-3)(12k-5) = 0$

$(4k-3) = 0 \; or \; (12k-5) = 0$

ie $k = \frac{3}{4} \; or \; \frac{5}{12}$

The conclusion is that wherever the smaller side is $\frac{5}{12} \; or \; \frac{3}{4}$ of the bigger side, the Bhodhayan concept gives the correct answer and in all other cases, it gives wrong answer.

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DoB & Place of Birth : 30.05.1954 ; Sokkankudiyiruppu, Tuticorin Dist., Tamilnadu
Parents : K. Michael & M. Soosai Ammal
Education : B.B.A. (Bachelor of Business Administration)

Employment Details :

1976 – 1978 : Tamilnadu Secretariat Service, Chennai
1978 - 1996 : Inspector of Customs & Central Excise, Madurai Commissionerate
1996 –2013 : Superintendent of Central Excise, Madurai & Tirunelveli Commissionerates
2013 - 2014 : Assistant Commissioner of Customs, Central Excise & GST, Tuticorin. (Retired on 31.05.2014)

Educational Service:

In 1995, founded the CEOA Educational Society which is at present blessed with seven Schools and one College located in southern Tamil Nadu, catering to 10000 plus students in all. Currently the Chairman of the CEOA Educational Society.

Books Authored:

தமிழ்

- விருத்த மணல் மாதா (1978)
- குறள் மோனை (2010)
- காணாமல் போன கலைச் சொற்கள் (2011)
- இனிக்கும் தமிழ் (2020)
- எம்மொழியே செம்மொழி (2023)

Maths

- The Geometry of Concurrency - V - 1 (2018)
- The Geometry of Concurrency - V -2 (2019)
- The Novelties Of Geometry (2023)
- Raja Climax's Theorem on Geometry (2023)
- Advanced Theorems on Geometry (2024)

Other Designations (From 1976)

- Ex- General Secretary, Madurai Central Excise Executive Officers' Association
- Ex - General Secretary, Tirunelveli Central Excise Gazetted Executive Officers' Association
- Founder Chairman, CEOA Group of Institutions
- Tamil Ilakana Vallunar
- Economic Analyst
- Globally renowned Geometer in Mathematics

Titles Earned

- Thaniththamil Navalar (Thiruvalluvar Mandram, Madurai)
- Muthamil Vendar (Ilango Muthamil Mandram, Madurai)
- Ilakkanachemmal (Thiruvalluvar Mandarm, Madurai)
- Honourary Doctorate (University of Asia)
- Ekalavya Award [Association of International Mathematics Education & Research (AIMER), Andhra Pradesh]

Lyrics written:

"Poopothigai" (youtube link : https://youtu.be/zTjZFutBaCY)
"Pothigai Tamzhil" (youtube link : https://youtu.be/hS0tLi07fDY)
"Pulavane" (youtube link : https://youtu.be/P-90V_NanEg)

www.maxgeometricmaths.co.in

(A website for Geometry lovers)

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Prize winners will be announced on 15th of the month, at 6:00 PM IST.

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